# Monte Carlo Method for the Evolution of Star Clusters Lecture IV

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## Star Cluster Dynamics and Evolution

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# Basic Assumptions

As we already learned from the previous lecture, to solve the Fokker-Planck equation, we need additional assumptions that simplify the problem and make it feasible. These are usually assumptions that strongly reduce the dimensionality of the problem, e.g.: equal mass, homogeneous cluster, isotropic velocity dispersion, spherical cluster. All those assumptions make the system non-physical but give qualitatively picture of the system structure and evolution.

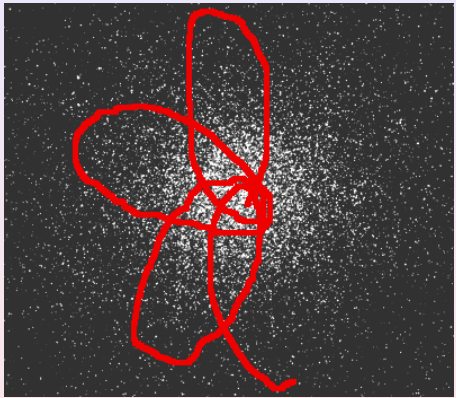
In the previous part of this lecture, we learned about the direct attack method, the N-body method, a method that relies on "brute" force to solve the equations of motion of all stars simultaneously. We learned about all its advantages and disadvantages.

- no additional assumptions regarding the star system - any structures and any environment
- brute force to calculate the force acting between the stars - various techniques are used to speed up codes: e.g. division on regular and irregular forces, parallelization ...
- relatively easy way in adding additional physical processes into the numerical code
- simulation time proportional to $N^4 (\sim N^2 log(N))$
- problems with simulation scaling to large N

# Basic Assumptions

**Monte Carlo method uses theoretical knowledge about system structure and evolution time scales. This is often the method of choice for solving partial differential equations**

- Spherical symmetry - quick and easy potential computation in any place in the system.
- Dynamical equilibrium - star cluster is old, its age is much greater than the evolution time scale - no violent relaxation.
- In specified potential, every star is characterized by its mass $M$, energy $E$ and angular momentum $J$ and moves on a rosette orbit - easy way o picking up the new object position.
- Evolution is driven by the relaxation process. The evolution time scale of each place in the system is proportional to the local relaxation time.



# Basic Tricks

- Since $N$ is large, the gravitational field can be divided into two parts: the averaged field and the field characterized by small irregular fluctuations. The motion of the star over the time interval $t_c < \Delta t < t_{rh}$, to a first approximation, is determined by the mean field, which is spherically symmetric and varies on the time scale $t_{rh}$ - it is independent of time on the $\Delta t$ scale. The orbit of a star is described by analytical formulas and there is no need to integrate the motion of the stars!
- However, the fluctuating field, although small, cannot be completely neglected. It randomly changes the parameters of the orbits. The effect of changes over time $\Delta t$ is small, but accumulates and becomes significant over time $t_{rh}$. Necessary is to take into account the influence of all stars on each point of the orbit at time $\Delta t$. Back to direct integration of the equations of motion!?

# Basic Tricks

Statistical way of solving the Fokker-Planck equation.

- Instead of integrating a sequence of uncorrelated small–angle perturbations along the orbit, a single perturbation is computed at a randomly selected point on the orbit.
- Instead of considering the effect of all stars in the system, the perturbation is computed locally from a randomly chosen star, in practice the nearest neighbor
- The computed single perturbation is multiplied by an appropriate factor in order to account for the cumulative effect of all small individual encounters with the rest stars in the system and for other points on the orbit

If the procedure is correctly set up, the evolution of an artificial system is statistically the same as the evolution of a real one.

# Basic Tricks

From simple two-body theory it is possible to calculate the change of the kinetic energy of a test star

$m_1(\Delta \mathbf{v_1})^2 = 4\frac{m_1 m_2^2}{(m_1+m_2)^2} w^2 sin^2(\frac{\beta}{2}), \quad$ where $\quad w = |\mathbf{v_{1o}} - \mathbf{v_{2o}}|$

According to the relaxation theory the average change of the kinetic energy of a test star is:

$\langle m_1(\Delta \mathbf{v_1})^2 \rangle = 8\pi G^2 n \Delta t \langle m_1 m_2 w^{-1} \rangle ln(\gamma N)$

Equating the above equations, we get:

$sin^2(\frac{\beta}{2}) = 2\pi G^2 \frac{(m_1+m_2)^2}{w^3} n \Delta t ln(\gamma N) \qquad (1)$

Similarly for the second moment:

$sin^4(\frac{\beta}{2}) = \pi G^2 \frac{(m_1+m_2)^2}{w^3} n \Delta t$

Finally

$sin^2(\frac{\beta}{2}) = \frac{1}{2ln(\gamma N)}$

Eq.1 describes the evolution of the system due to the relaxation process. It connects evolution time step, local system properties with the changes of the star velocities

Basic free parameters of the Monte Carlo method

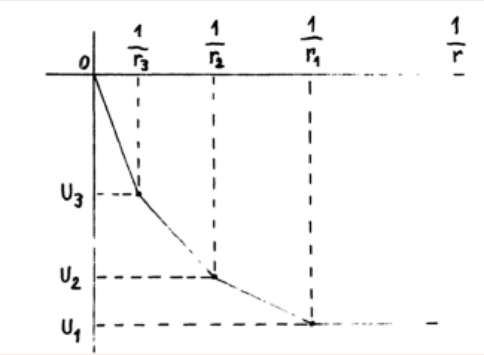
- deflection angle, $\beta$ - controls the size of the time step $\Delta t$. It has to be very small, according to the relaxation theory.
- time step $\Delta t$ - fraction of the local relaxation time and larger than the local crossing time
- $\gamma$, coefficient in the Coulomb logarithm. It needs to be determined by comparison with results of N-body simulations

# Basic Ingredients

- The Monte Carlo code is based on Hénon's (1971) implementation of the Monte Carlo method, which was substantially further developed by Stodółiewicz (1982, 1983, 1986) and by Giersz (1998 - ....)
- A star cluster is treated as a set of spherical shells which represent an individual object star, binary ... or a group of the same objects. Each shell is characterized by: mass (m), energy (E) and angular momentum (J)
- Relaxation process of a given object with all other objects in the system is approximated by only ONE interaction of two neighboring shells - Hénon's trick
- The whole system is divided on **superzones**. Each superzone is characterized by its own time step $\Delta t$. Time steps for successive superzones are larger by factor 2 - **Block Time Step Scheme**

# Basic Ingredients - Smooth Potential

The mean smooth potential is computed as potential of concentric spherical shells




Let's assume that $r_0 = 0$ and $r_n = \infty$. The potential in a shell between k and k+1 is

$$U(r) = G\left(-\frac{1}{r}\sum_{i=1}^{k} m_i - \sum_{i=k+1}^{n}\frac{m_i}{r_i}\right)$$

Knowing potential in each shell $U_k = U(r_k)$ it is possible to compute potential in any position $r$

$$U(r) = U_k + \frac{1/r_k - 1/r}{1/r_k - 1/r_{k+1}}(U_k - U_{k+1}), \quad r_k \le r \le r_{k+1}$$

Only $N$ operation is needed to compute potential

# Basic Ingredients - New Position

A star describes a rosette orbit with $r$ oscillating between two extreme values $r_{min}$ and $r_{max}$. $r_{min}$ and $r_{max}$ are the roots of the energy equation

$$2E - 2U(r) - \frac{J^2}{r^2} = 0$$

After relaxation step we need to select new position between $r_{min}$ and $r_{max}$

The probability to find a star in a new position in interval $dr$ is proportional to the fraction of time spent by a star in $dr$

$$\frac{dt}{P} = \frac{dr/\mid v_r \mid}{\int_{r_{min}}^{r_{max}} dr/\mid v_r \mid}$$

$$\mid v_r \mid = \left(2E - 2U(r) - \frac{j^2}{r^2}\right)^{1/2}$$

$P$ is the orbital period

# Monte Carlo Scheme

The relaxation step in the Monte Carlo method consists

- sorting all zones by position, $r$
- calculation of deflection angles and new velocities for each pair of zones - relaxation for time step $\Delta t$
- finding the new positions for every pair of zones
- after taking into account all zones return to the first point

Computation of the star cluster evolution due to the relaxation process is only the first step. To compute evolution of real star cluster we need to take into account additional physical processes connected with stellar/binary evolution, binary interactions and formation, tidal field and escape of stars ... This has to be done not only for the Monte Carlo method but also for other methods, including N-body method

The description of those processes will be the subject of the next Lectures