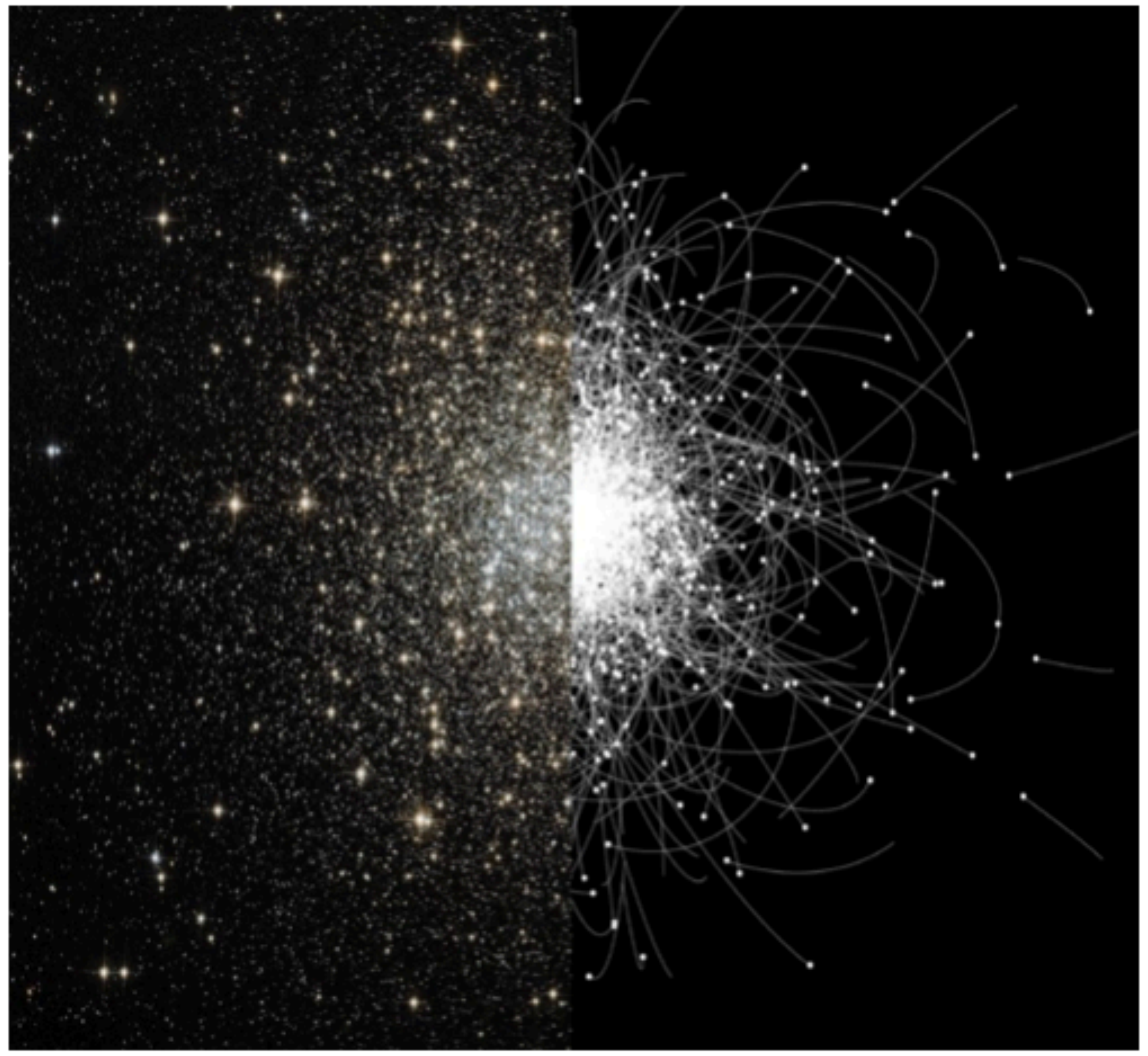
Star Cluster Dynamics and Evolution



Geoplanet Doctoral School Lecture Course (Spring 2024)

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Outline: *N*-body simulations of gravitational dynamics

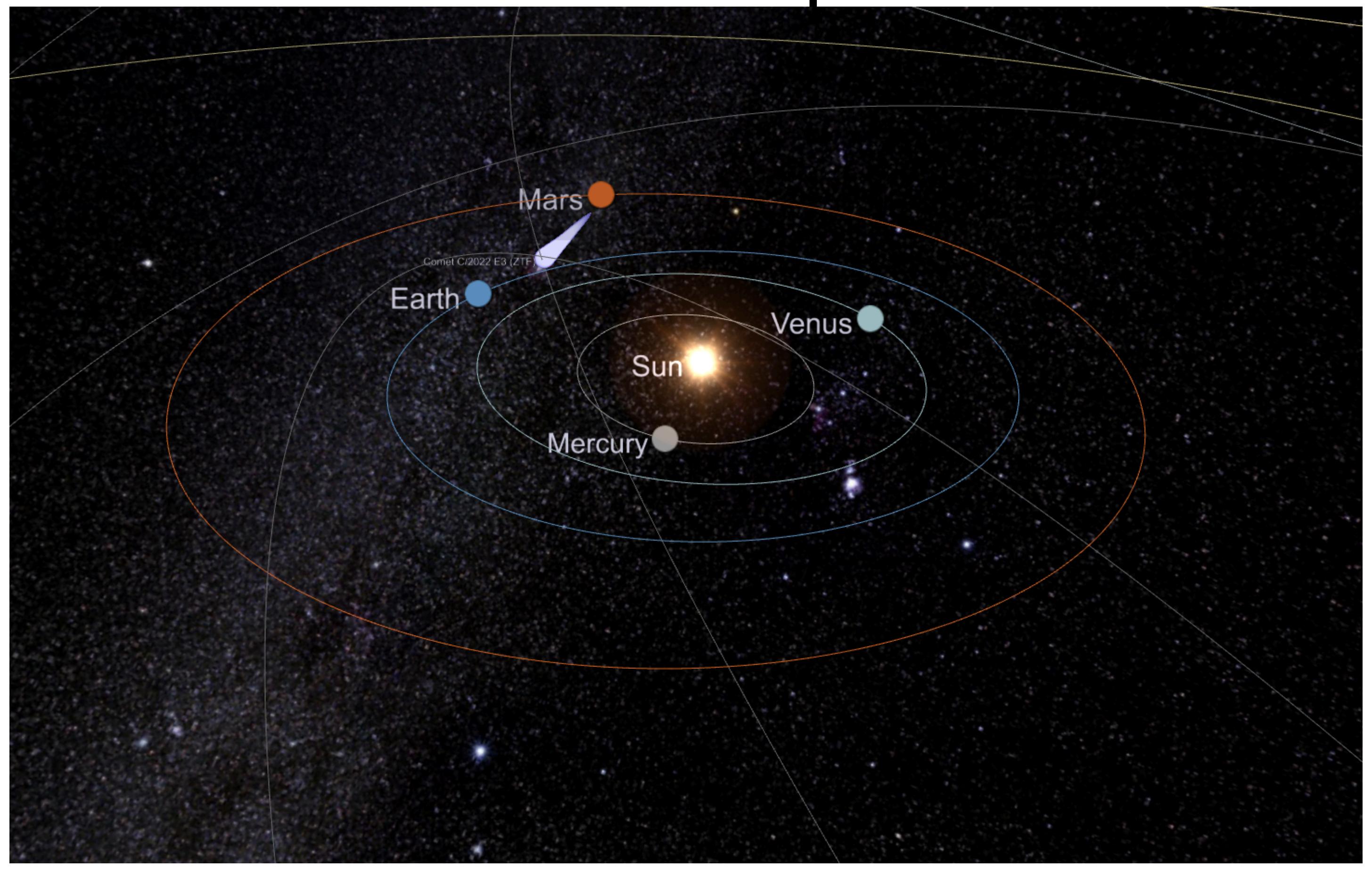
- The *N*-body problem and its astrophysical settings
  - What is an *N*-body system?
- Solving the equations of motion
  - Complexities and challenges
- Structure of a simples direct-summation *N*-body code
  - Integration strategies
  - Timesteps
  - Regularization

Classical gravity and the *N*-body problem

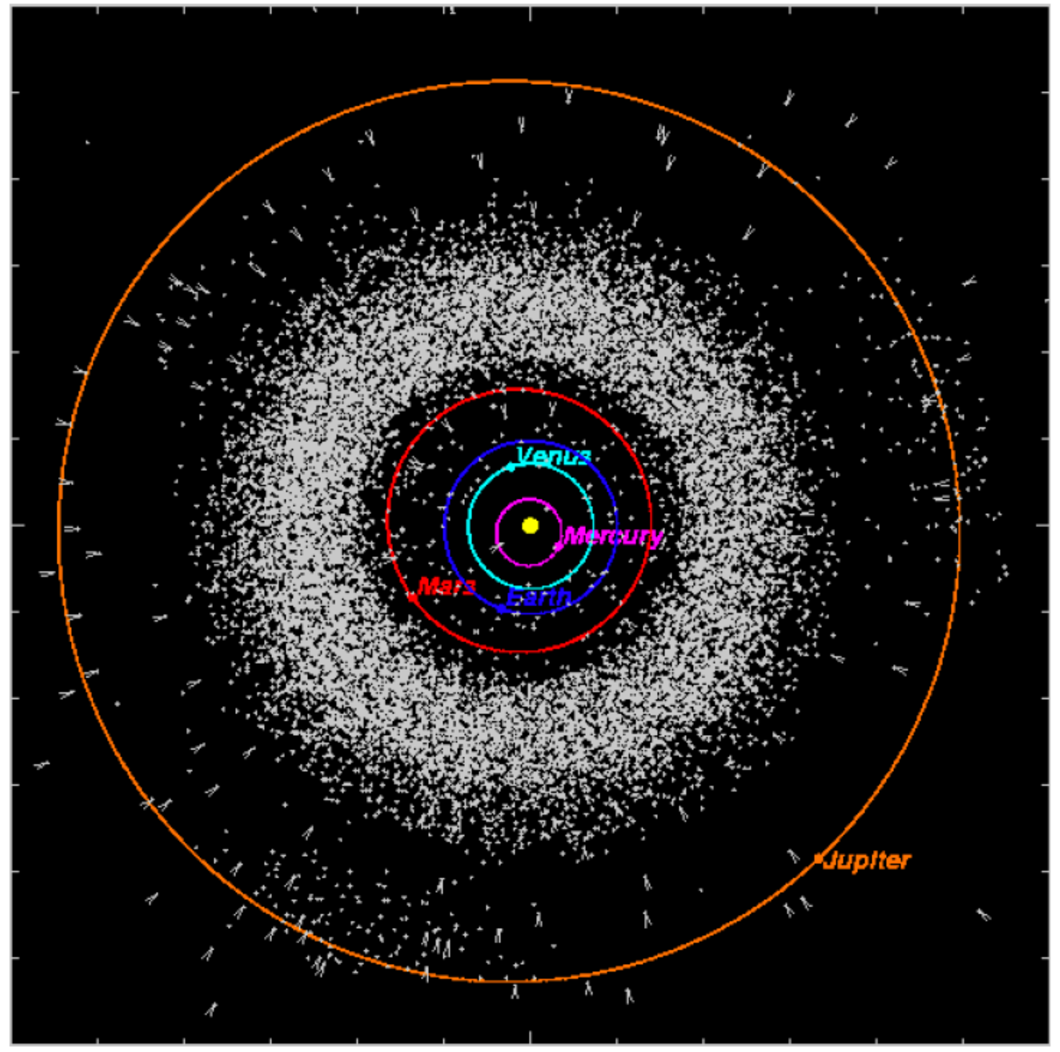
- Predicting the individual motions (position and velocities) of a group of objects (*N*=1,2,3,...) interacting due to gravity
- Newton formulated the law of universal gravitation (1686)

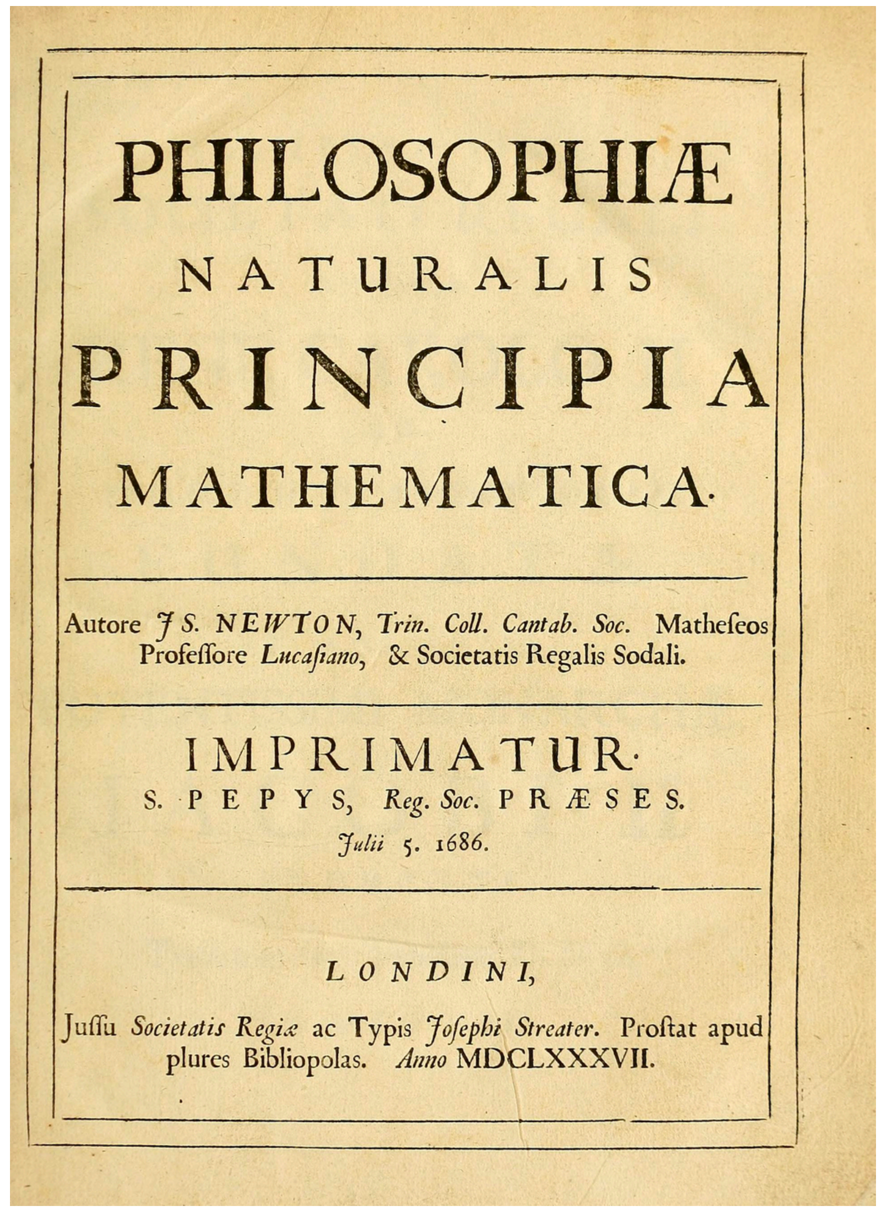
$$F = G\frac{m_1 m_2}{r^2}$$

  - Was motivated, at least in part, by a desire to understand the movements of the planets



Credit: The Sky; 3D Solar System Simulator

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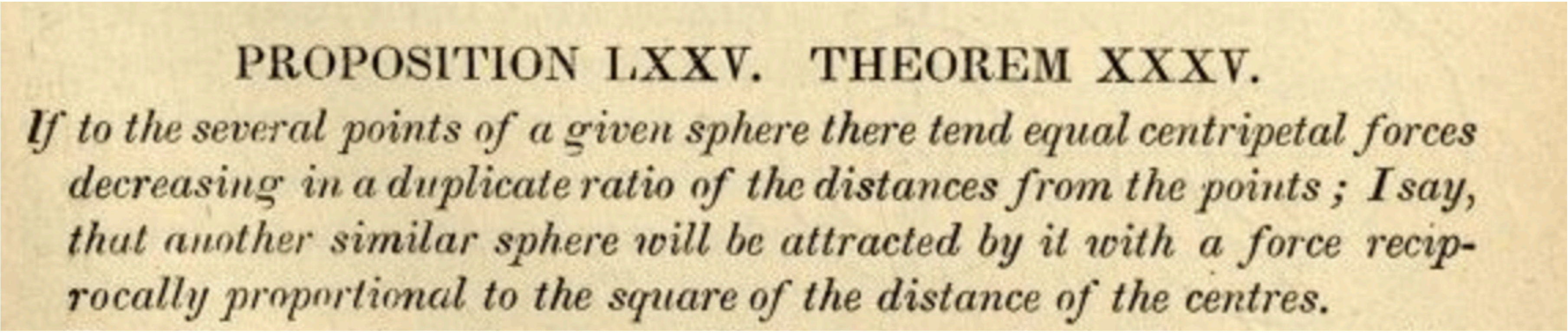
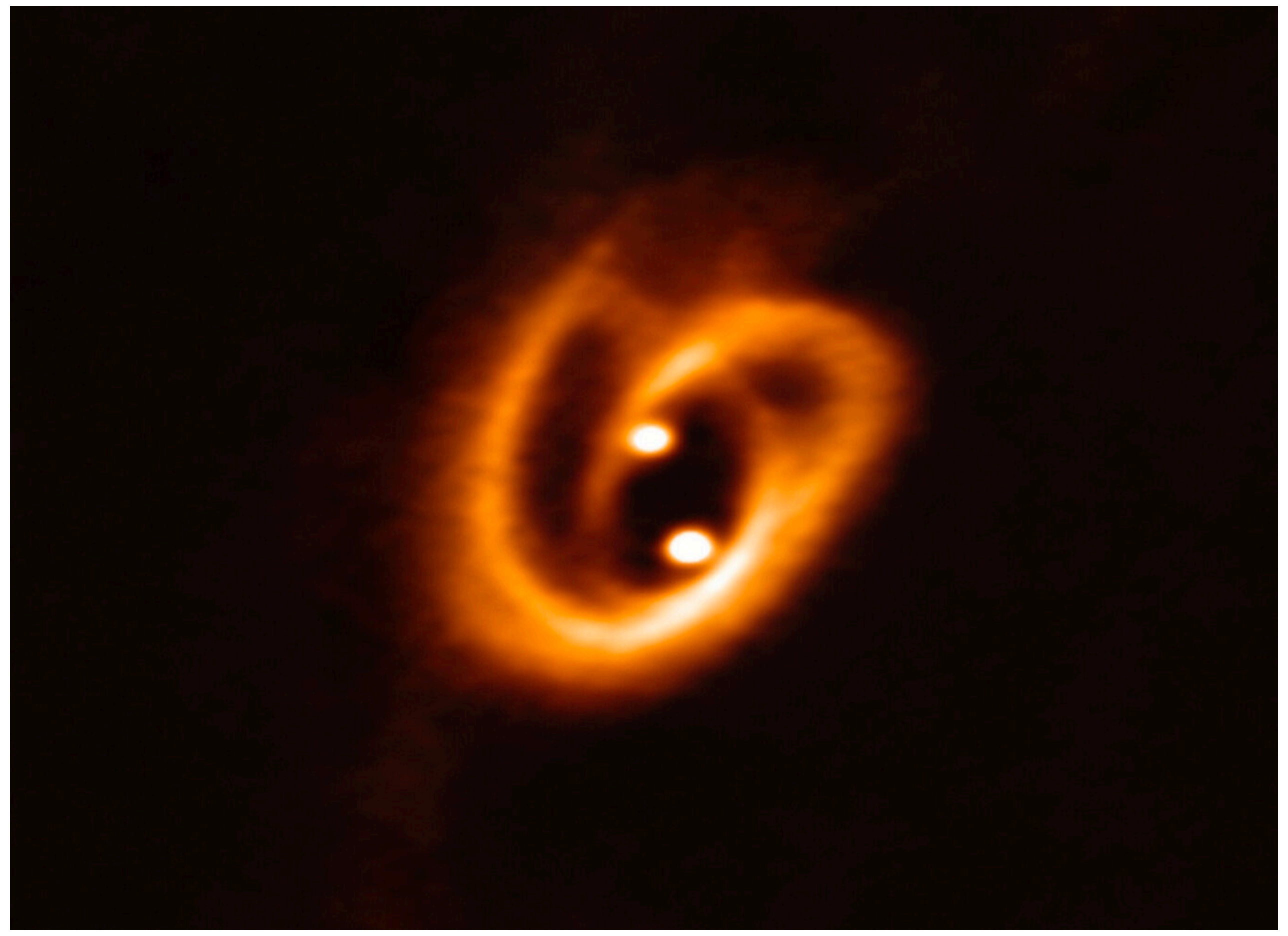

PROPOSITION LXXV. THEOREM XXXV.

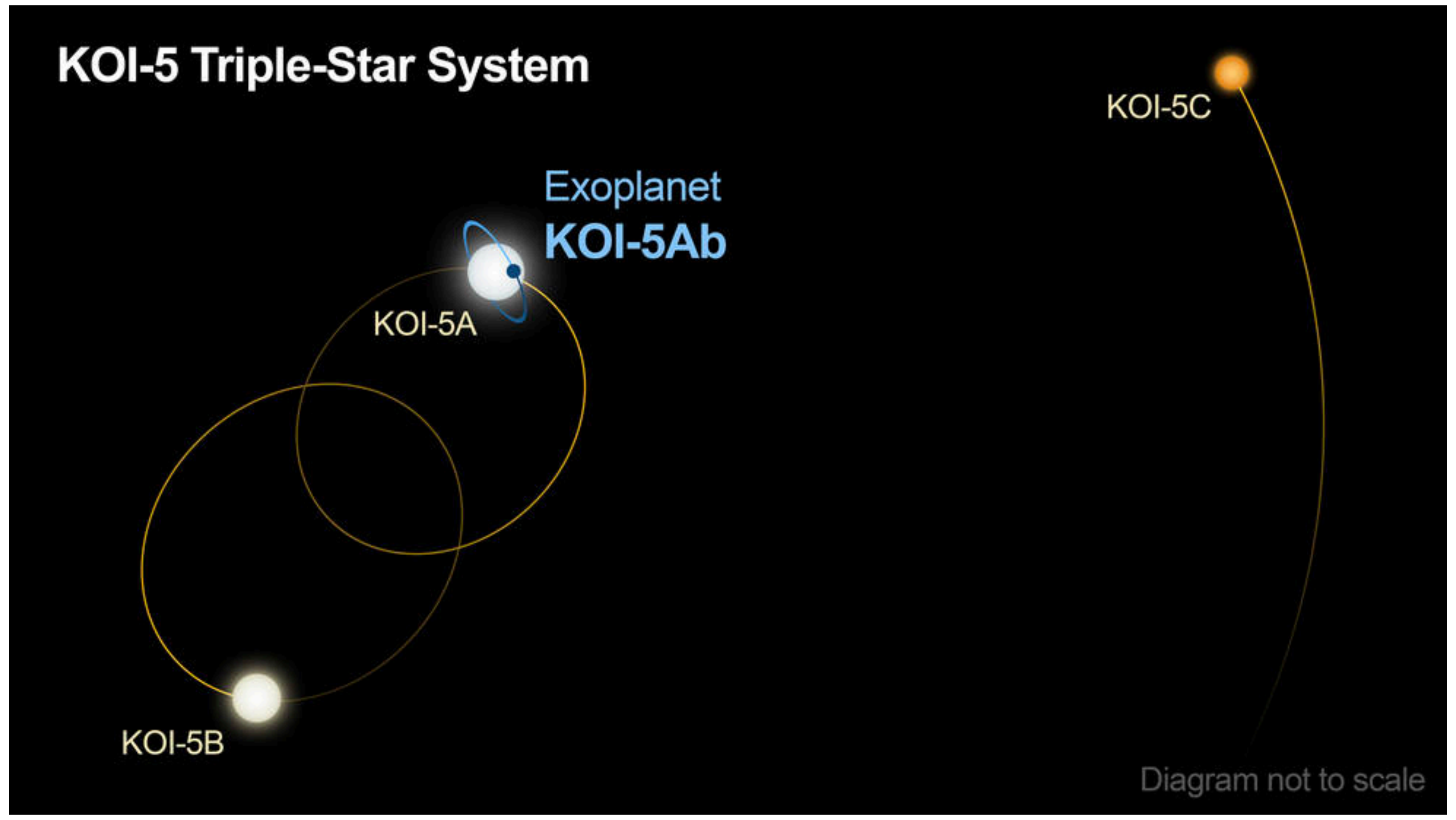
If to the several points of a given sphere there tend equal centripetal forces decreasing in a duplicate ratio of the distances from the points; I say, that another similar sphere will be attracted by it with a force reciprocally proportional to the square of the distance of the centres.

Credit: UC Libraries' Internet Archive

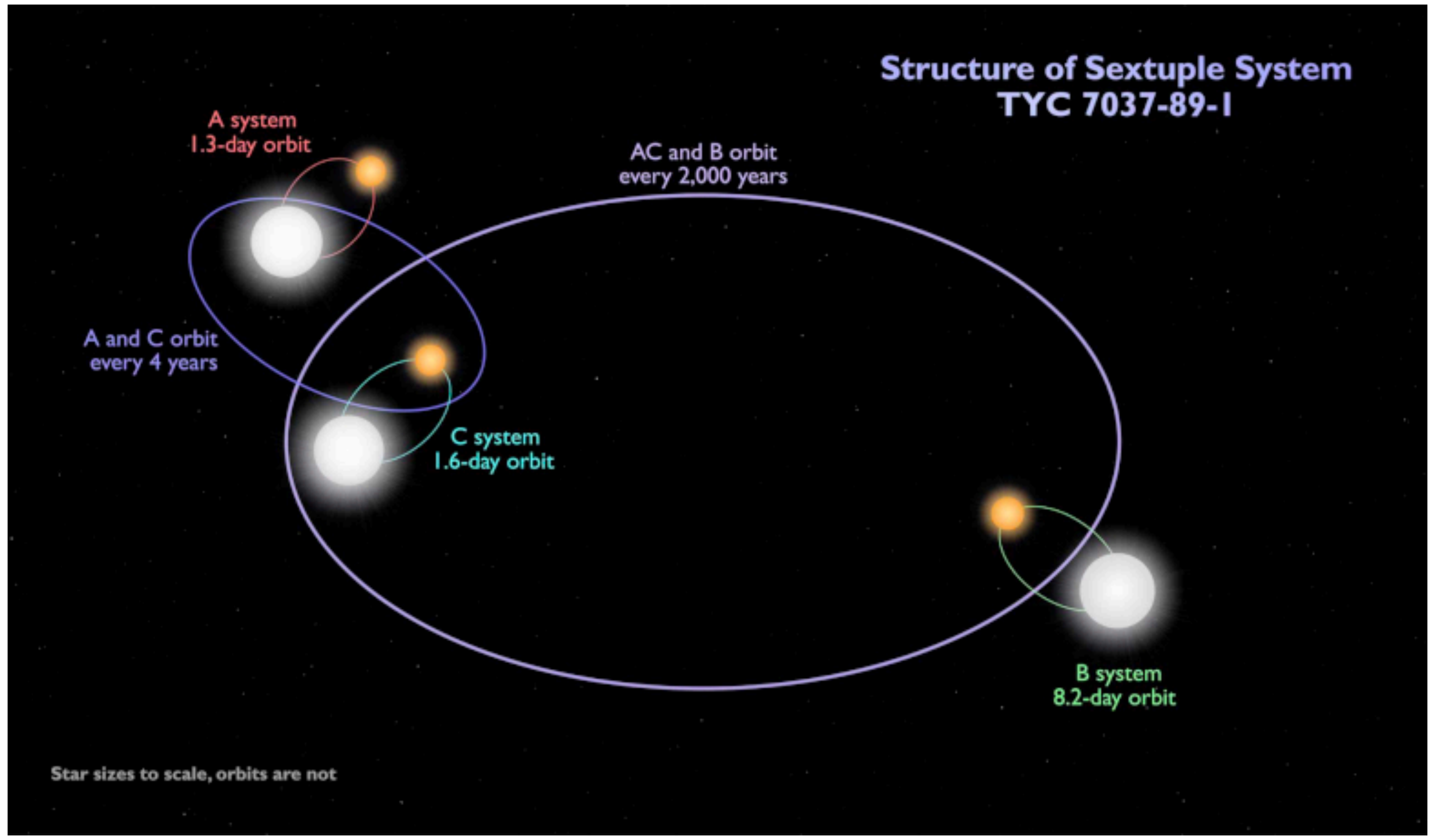
Astrophysical *N*-body systems: Binary systems, triples and multiples



Binary Stars
Credit: ALMA (ESO/NAOJ/NRAO),
Alves et al. 2019

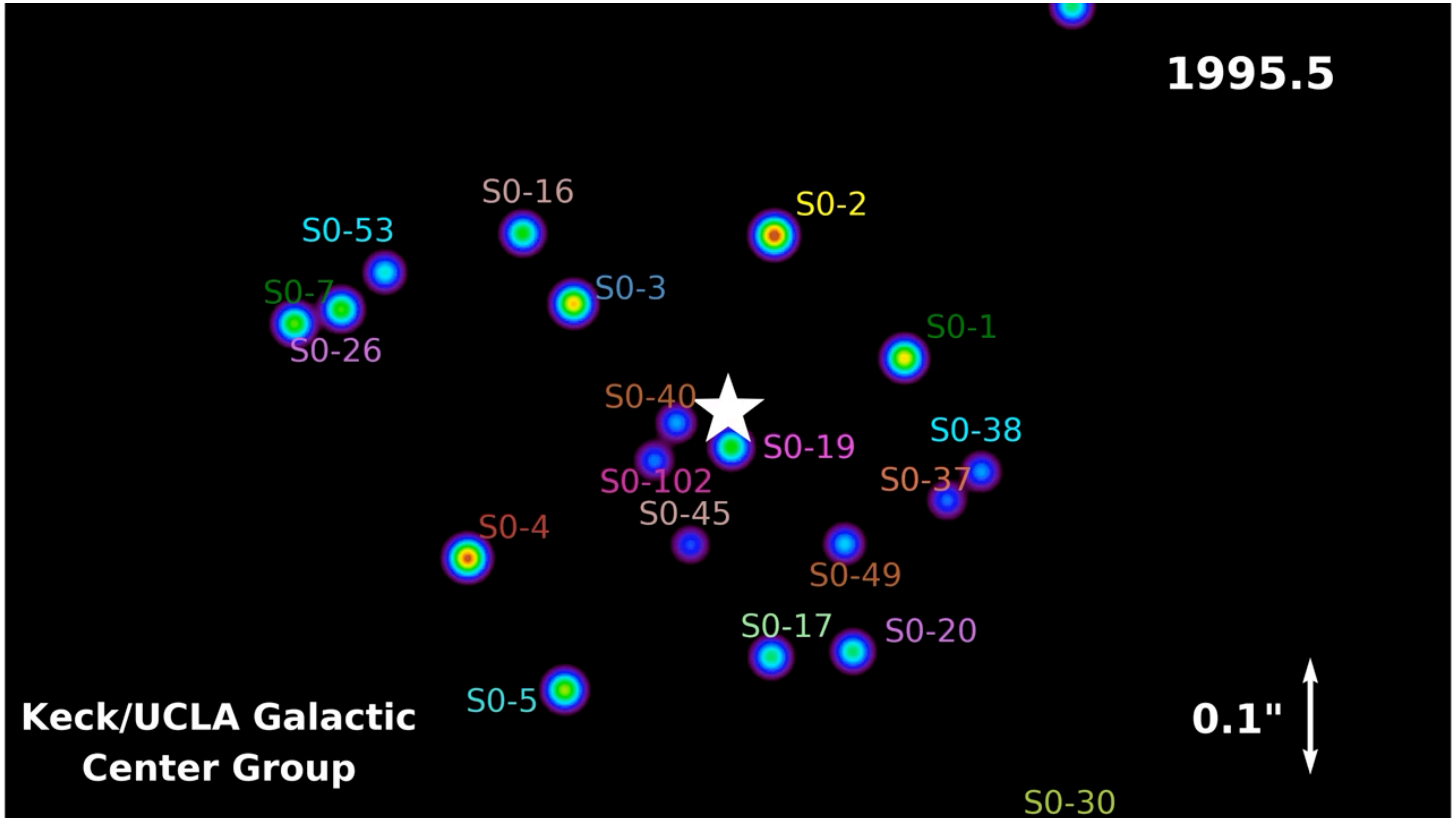


KOI-5 star system
Credis: Caltech/R. Hurt (IPAC)



TYC 7037-89-1; a triple-binary
sextuple star system
Credit: GSFC

Astrophysical *N*-body systems: The Galactic Center



Orbit of stars around Sgr A* in the Galactic Center
Credit: Keck/UCLA

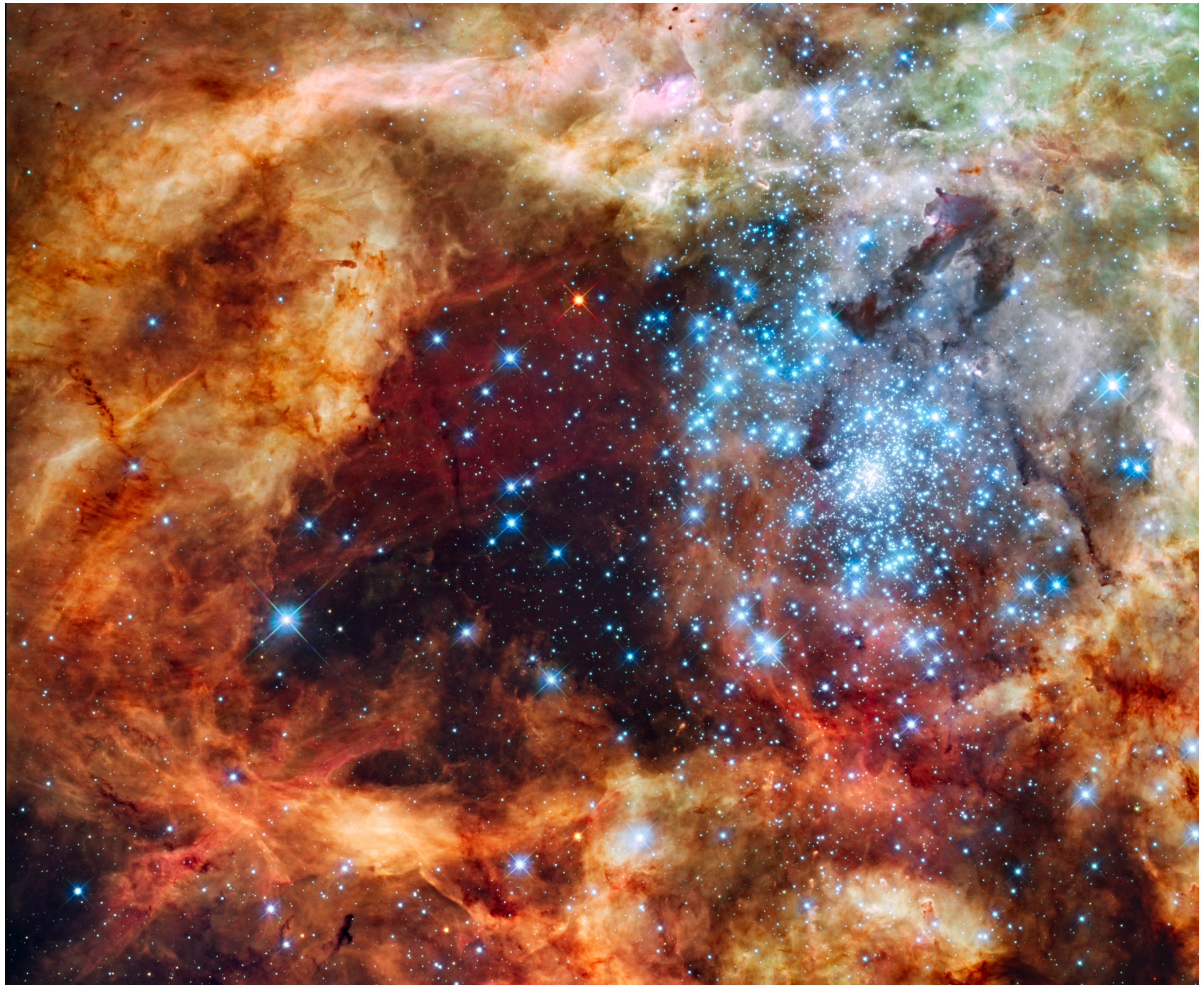
Astrophysical *N*-body systems: Star Clusters

Open Cluster M67
Credit: Jim Mazur's Astrophotography

Globular Cluster M4
Credit: ESO La Silla 2.2m telescope

Astrophysical *N*-body systems: Young Star Clusters

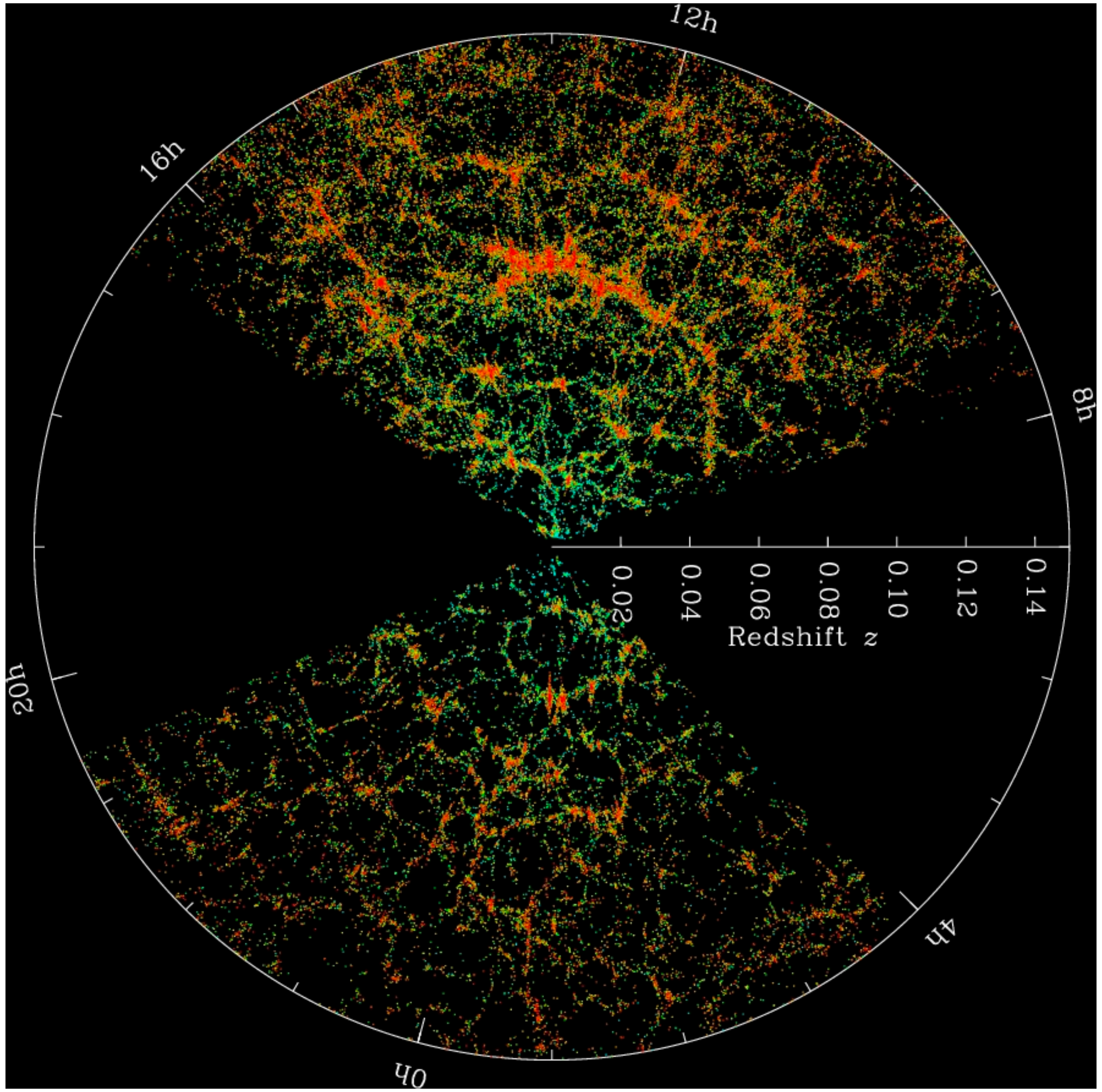
NGC 1850 in the Large Magellanic Cloud
Credit: NASA/ESA *Hubble Space Telescope*



R136 star cluster in the LMC
Credit: NASA/ESA *Hubble Space Telescope*

Astrophysical *N*-body systems: Galaxy dynamics and cosmic structures

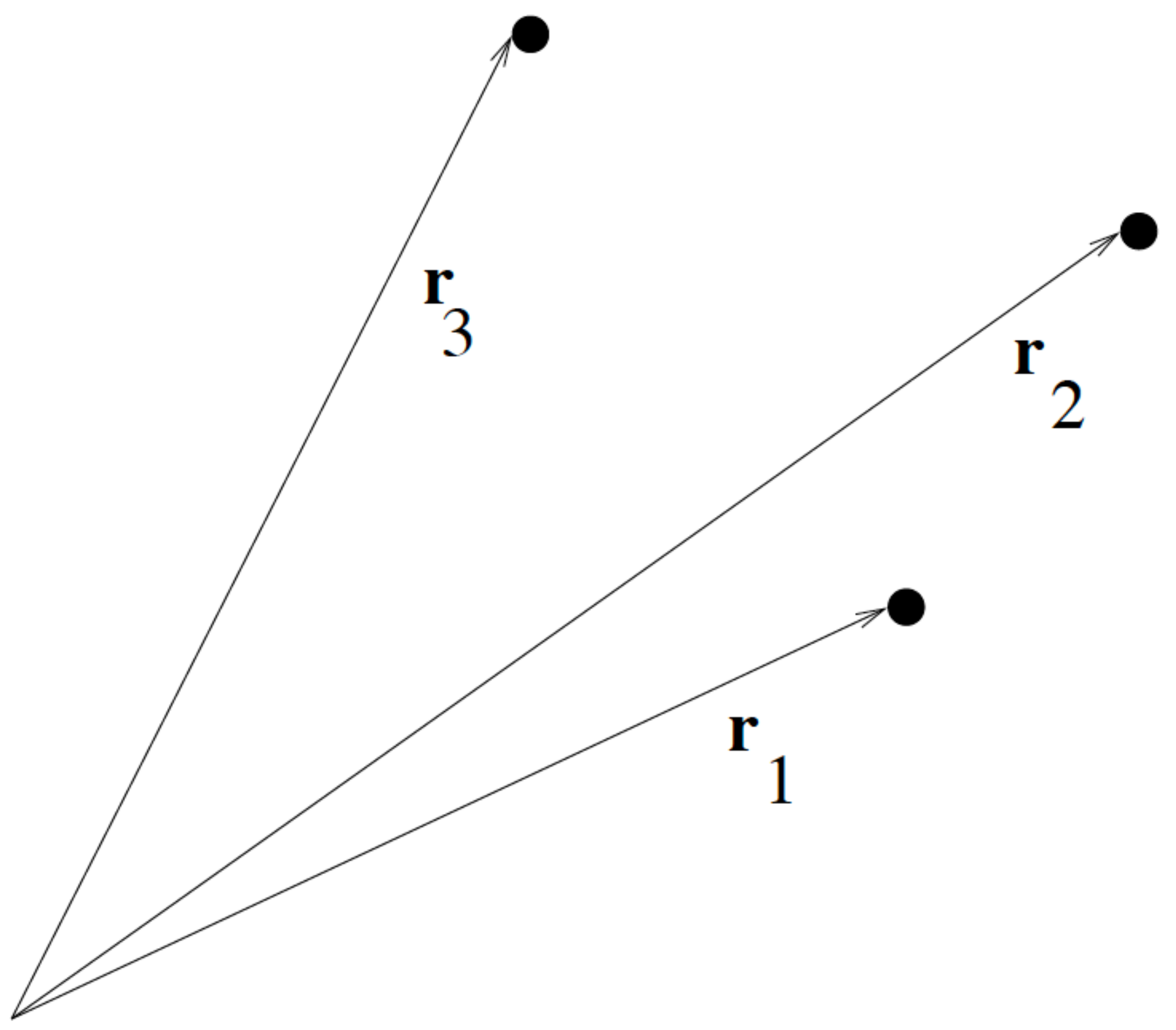
Antennae Galaxies
Credit: NASA/ESA *HST*



Each dot is a galaxy
Credit: Sloan Digital Sky Survey (SDSS) map of the Universe

The gravitational N-body problem: Equation of motion

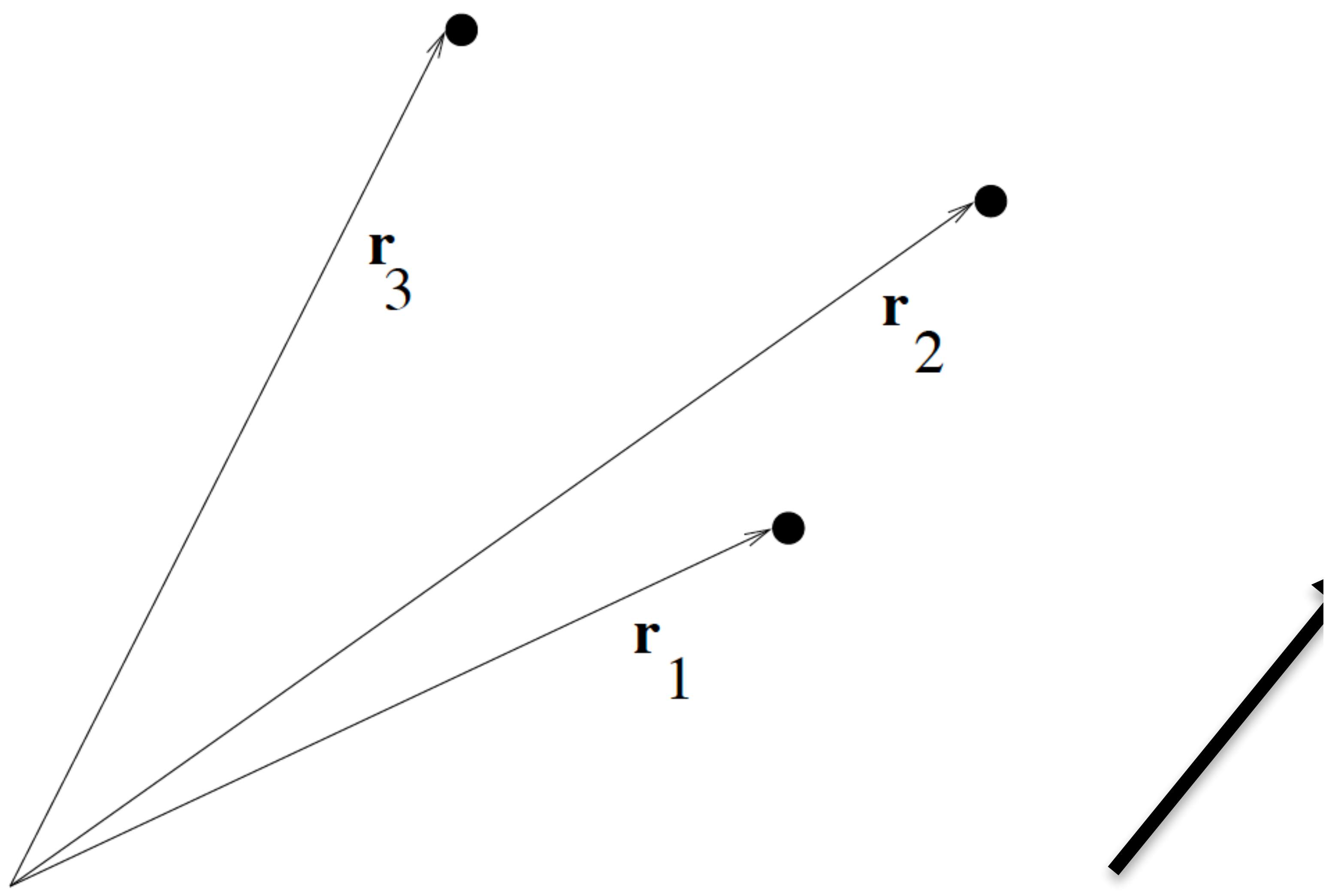
Equations of motion



Force on 1 due to 2 is in the direction of $\mathbf{r}_2 - \mathbf{r}_1$, i.e. the unit vector $\dfrac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$, and has magnitude $\dfrac{Gm_1 m_2}{|\mathbf{r}_2 - \mathbf{r}_1|^2}$.

The gravitational *N*-body problem: Equation of motion

Equations of motion



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Therefore force on 1 due to 2 is $\frac{Gm_1 m_2(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}$. Therefore total force on 1 is

$$m_1 \ddot{\mathbf{r}}_1 = \sum_{j=1, j\neq 1}^{j=3} \frac{Gm_1 m_j(\mathbf{r}_j - \mathbf{r}_1)}{|\mathbf{r}_j - \mathbf{r}_1|^3}$$

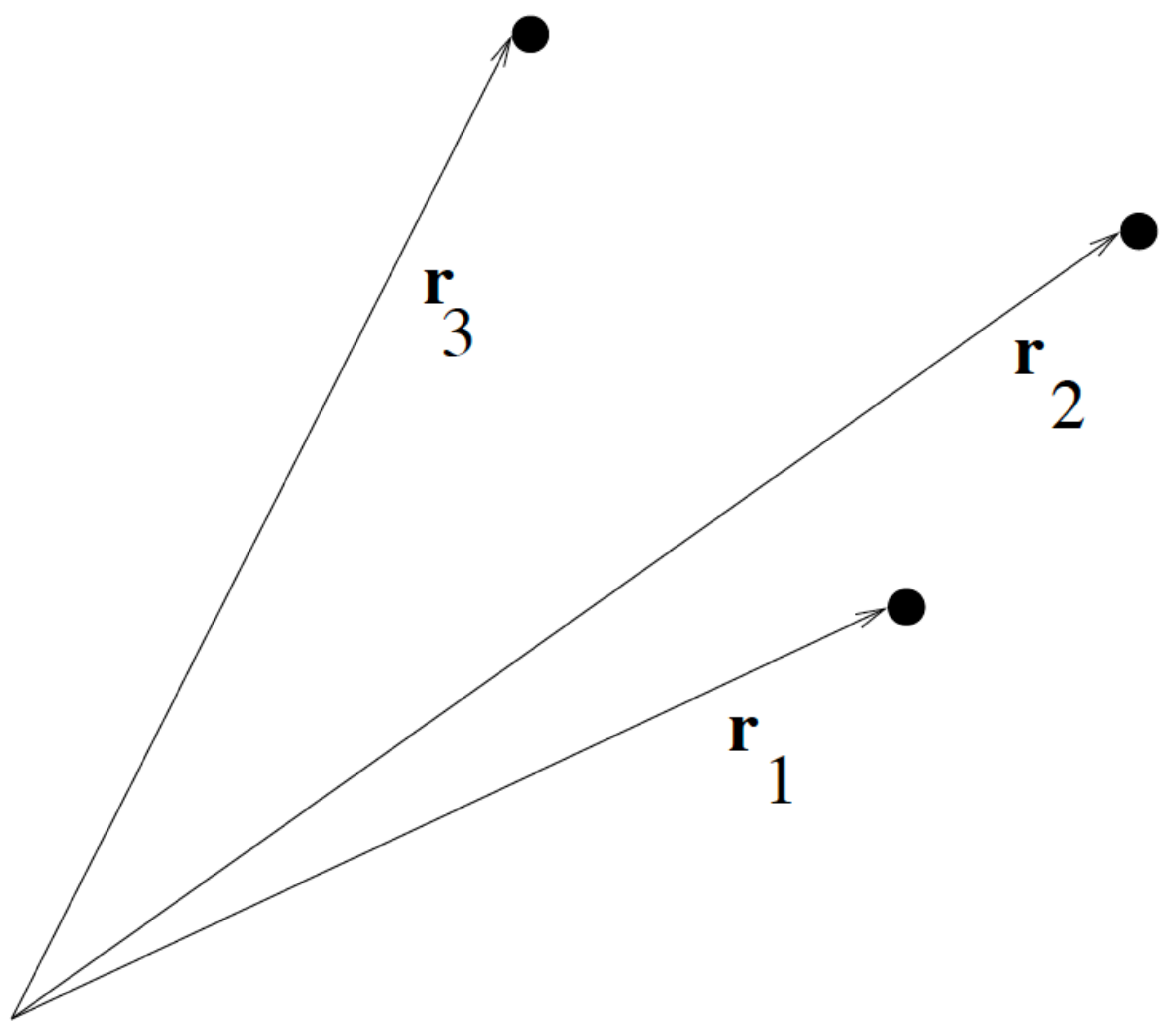
In the *N*-body problem the equation of motion for body *i* is

$$\ddot{\mathbf{r}}_i = -G \sum_{j=1,\neq i}^{N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

Stellar dynamics can be defined as studying the consequences of this equation in astrophysical contexts

The gravitational N-body problem: Equation of motion

Equations of motion



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Stellar dynamics can be defined as studying the consequences of this equation in astrophysical contexts

"These equations have an appealing — if deceptive —simplicity"
Lyman Spitzer, Jr

Complexities of gravity

- Analytic solutions only for N = 2 (and restricted N = 3)
- Force calculation is $\sim N \times N$ operation
- Gravitational force does not fade off (even far away particle interact)
  - Calculation of a system of *N* particles cannot be decomposed in smaller pieces
- At small distances the force of gravity grows limitless.
- The equations of motion are intrinsically chaotic
- Systems bound by gravity have a negative heat capacity
  - Our daily experience are not trained to appreciate the complexities of gravity

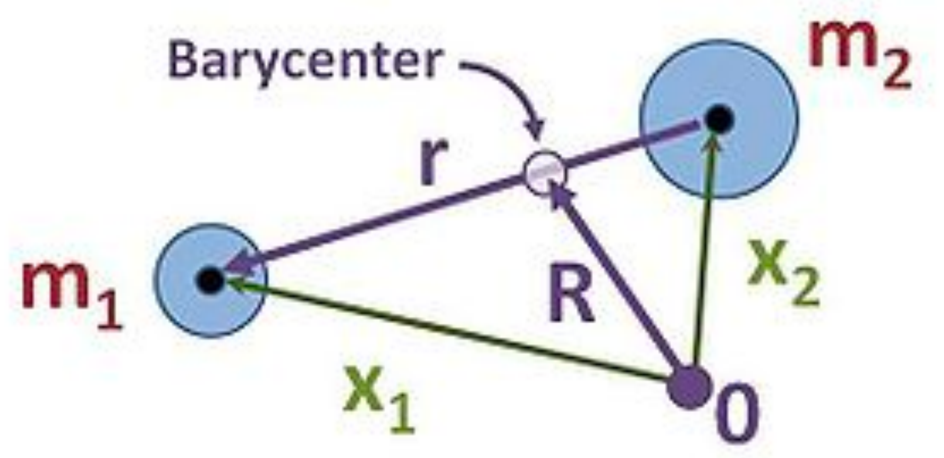
"Enormous range of length scales (and, consequently, time scales) is one reason why the *N*-body equations are a severe challenge to the computational [astro]physicist."

Heggie & Hut (The Gravitational Million Body Problem 2003)

Mathematics of the *N*-body problem

- For *N*=2 systems (two-body problem), solution can be computed analytically
  - e.g., Herman (1710) & Bernoulli (1710)
- Total momentum: $p = m_1 v_1 + m_2 v_2$
- Rate of change of momentum $\frac{dp}{dt} = m_1 \frac{dv_1}{dt} + m_2 \frac{dv_2}{dt} = F_1 + F_2$

$$F_1 = -F_2 \qquad \rightarrow dp/dt = 0 \rightarrow p = \text{ const.}$$



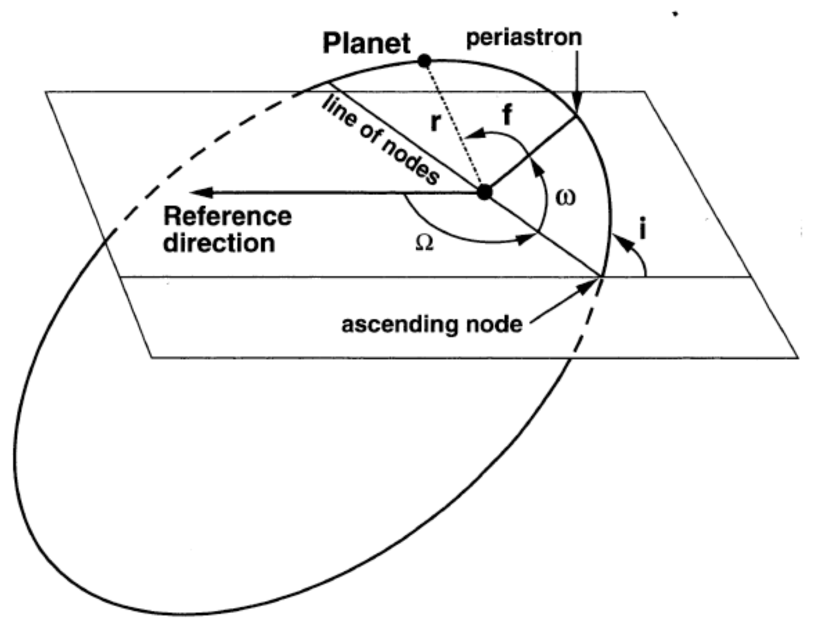


FIGURE 3.3 Diagram for an ellipse, illustrating the orbital elements Ω, ω, i, and f. The parallelogram is the reference plane inclined to the orbital plane by the angle i.

- Two bodies will move around a common point (CoM; center of mass) which moves at a constant velocity
- For *N*=3, there is no general analytic solution

1885: Prize competition in honour of King Oscar II

- A challenge was proposed (to be answered before 21/01/1889)

The *n*-body problem

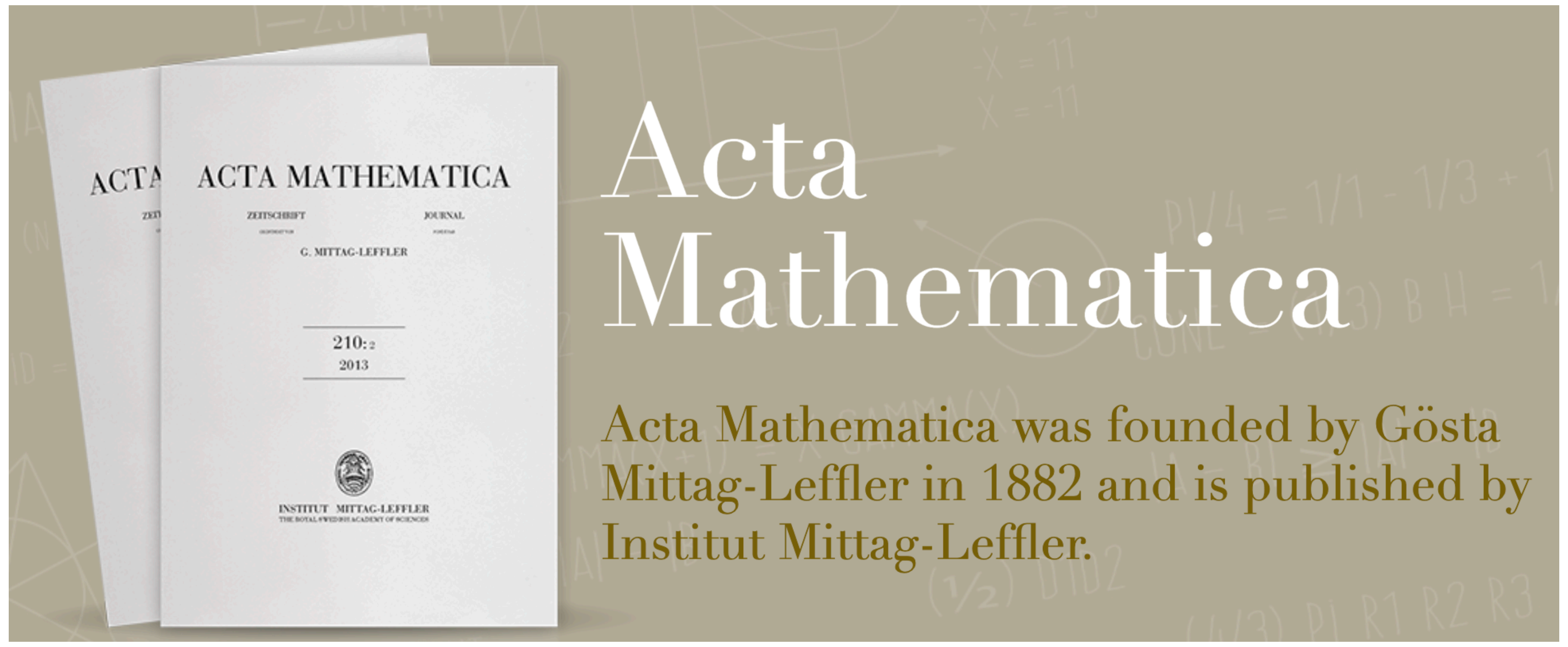
The first question was a formulation of the classical n-body problem in celestial mechanics, and it attracted the most attention:

> *For a system of arbitrarily many mass points that attract each other according to Newton's law, assuming that no two points ever collide, find a series expansion of the coordinates of each point in known functions of time converging uniformly for any period of time.*

Weierstrass nourished great hopes that there would be a method of solution based on a simple fundamental idea, and that such a solution could help conclude whether the system was stable. In applying this formulation of the n-body problem as a model of the solar system, could it be excluded that the planetary orbits might alter radically over a long time, or even that one planet be thrown out of the system?

http://www.mittag-leffler.se/library/henri-poincare

Credit: Hulton Archive



Mathematics of the *N*-body problem

- For *N*=3, there is no general analytic solution
  - Approximate solutions when one particle is much less massive than the other two
- Constraints
  - Total energy is conserved: $$E = \frac{1}{2}\sum_i m_i v_i^2 + \sum_i \sum_{j \neq i} \frac{G m_i m_j}{\left|\mathbf{r}_\mathrm{i} - \mathrm{r}_\mathrm{j}\right|} = \text{ const.}$$
  - Angular momentum is conserved: $$L = \sum_i \mathbf{r}_\mathrm{i} \times m_i \mathbf{v}_\mathrm{i} = \text{ const.}$$
  - Total momentum conserved: center of mass moves at constant velocity (6 more constraints)

$$\mathbf{V}_{CM} = \text{ const} \qquad \mathbf{X}_{CM} = \mathbf{X}_0 + \mathbf{V}_{CM} t$$

The gravitational *N*-body problem: Computation

$$\frac{d^2\mathbf{x}_i}{dt^2} = -\sum_{j=1;j\neq i}^{N} \frac{Gm_j(\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^3} \tag{1}$$

The description of the problem is completed by specifying the initial positions ($\mathbf{x}_i$ at $t = 0$) and velocities ($\mathbf{v}_i$ at $t = 0$) for the N particles. The N-body problem involves calculating

- the force on each particle at a given time;
- determining the new position of the particle at a future time.

- Motion of the particles given by Newton's law: $\frac{dX_i}{dt} = V_i \qquad m_i\frac{d\,V_i}{dt} = \mathbf{F_i}$
- Forces computed from Newton's law of gravity: $\mathbf{F_i} = \sum_{i\neq j} \frac{Gm_i m_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^2}$
- Direct *N*-body: Integrator to solve the ordinary differential equations (2 coupled ODEs)

Direct N-body technique: advantages

- The equation of motion of the *i*th star is

$$\ddot{\mathbf{r}}_i = -G \sum_{j=1,\neq i}^{N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^3}$$

- Numerically integrating these equations enables following the evolution of star cluster throughout its entire lifespan
- Least amount of assumptions, do not require:
  - Dynamical equilibrium
  - Non-rotating systems
  - Spherical symmetry

Holmberg's "*N*-body" experiment: Interacting galaxies

THE ASTROPHYSICAL JOURNAL

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ON THE CLUSTERING TENDENCIES AMONG THE NEBULAE

II. A STUDY OF ENCOUNTERS BETWEEN LABORATORY MODELS OF STELLAR SYSTEMS BY A NEW INTEGRATION PROCEDURE

ERIK HOLMBERG

ABSTRACT

In a previous paper[1] the writer discussed the possibility of explaining the observed clustering effects among extragalactic nebulae as a result of captures. The present investigation deals with the important problem of whether the loss of energy resulting from the tidal disturbances at a close encounter between two nebulae is large enough to effect a capture. The tidal deformations of two models of stellar systems, passing each other at a small distance, are studied by reconstructing, piece by piece, the orbits described by the individual mass elements. The difficulty of integrating the total gravitational force acting upon a certain element at a certain point of time is solved by replacing gravitation by light. The mass elements are represented by light-bulbs, the candle power being proportional to mass, and the total light is measured by a photocell (Fig. 1). The nebulae are assumed to have a flattened shape, and each is represented by 37 light-bulbs. It is found that the tidal deformations cause an increase in the attraction between the two objects, the increase reaching its maximum value when the nebulae are separating, i.e., after the passage. The resulting loss of energy (Fig. 6) is comparatively large and may, in favorable cases, effect a capture. The spiral arms developing during the encounter (Figs. 4) represent an interesting by-product of the investigation. The direction of the arms depends on the direction of rotation of the nebulae with respect to the direction of their space motions.

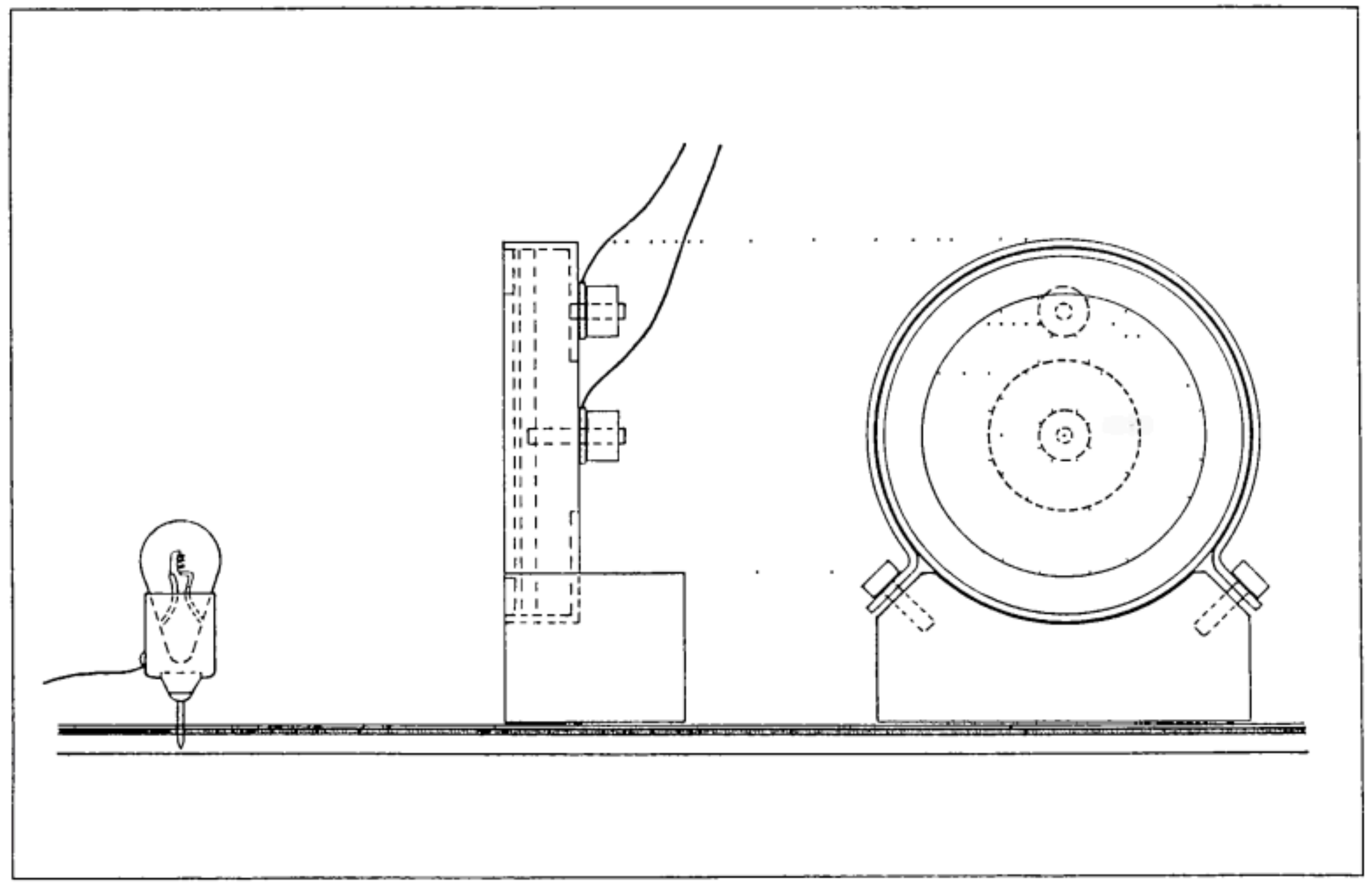

FIG. 1.—Cross-section of light-bulb and photocell (half-size)

Replaced gravity with light

$$\mathbf{I} \propto \left(1/\mathbf{d}^2\right)$$

Holmberg's "*N*-body" experiment: Interacting galaxies

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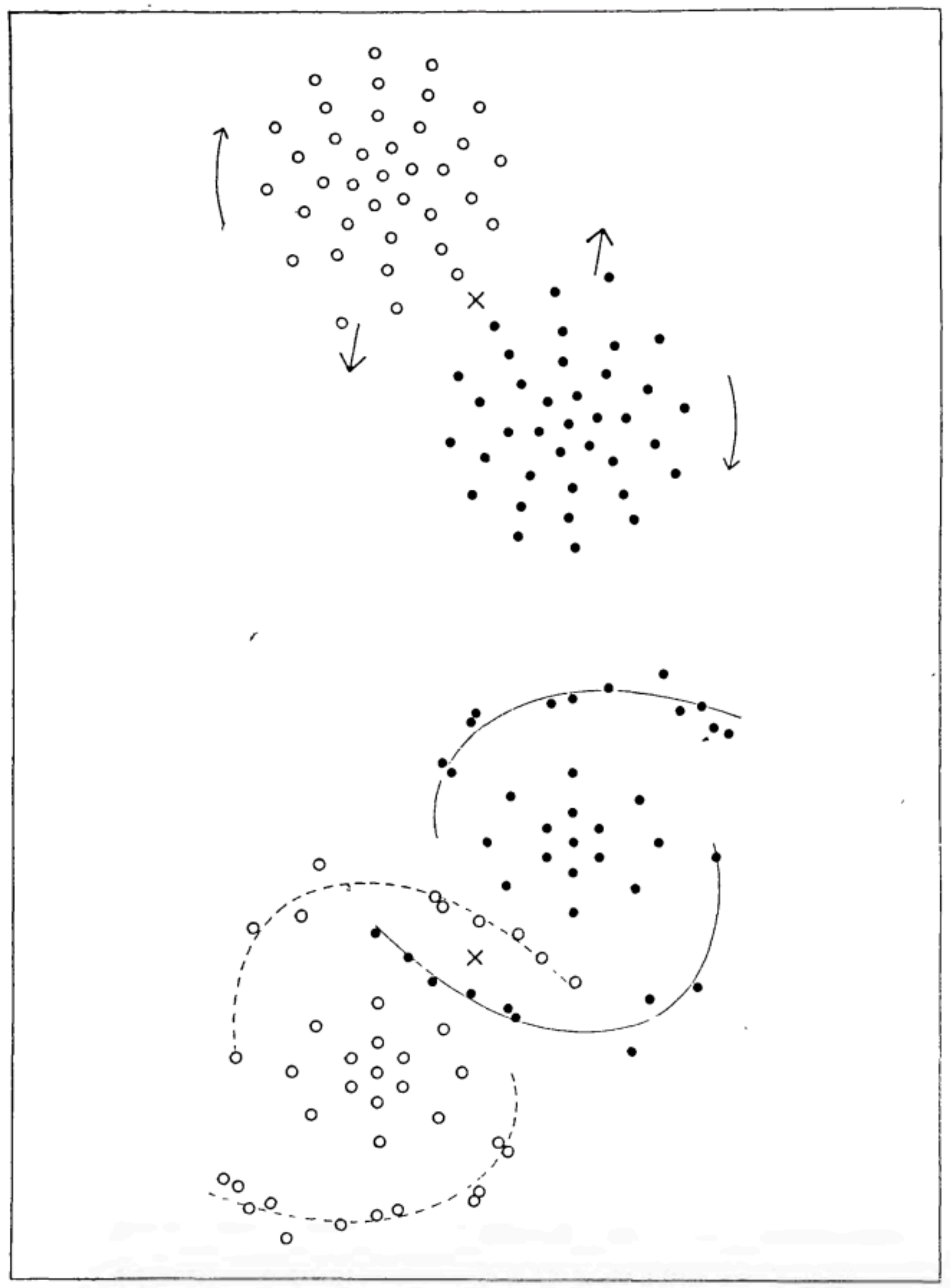

FIG. 4a

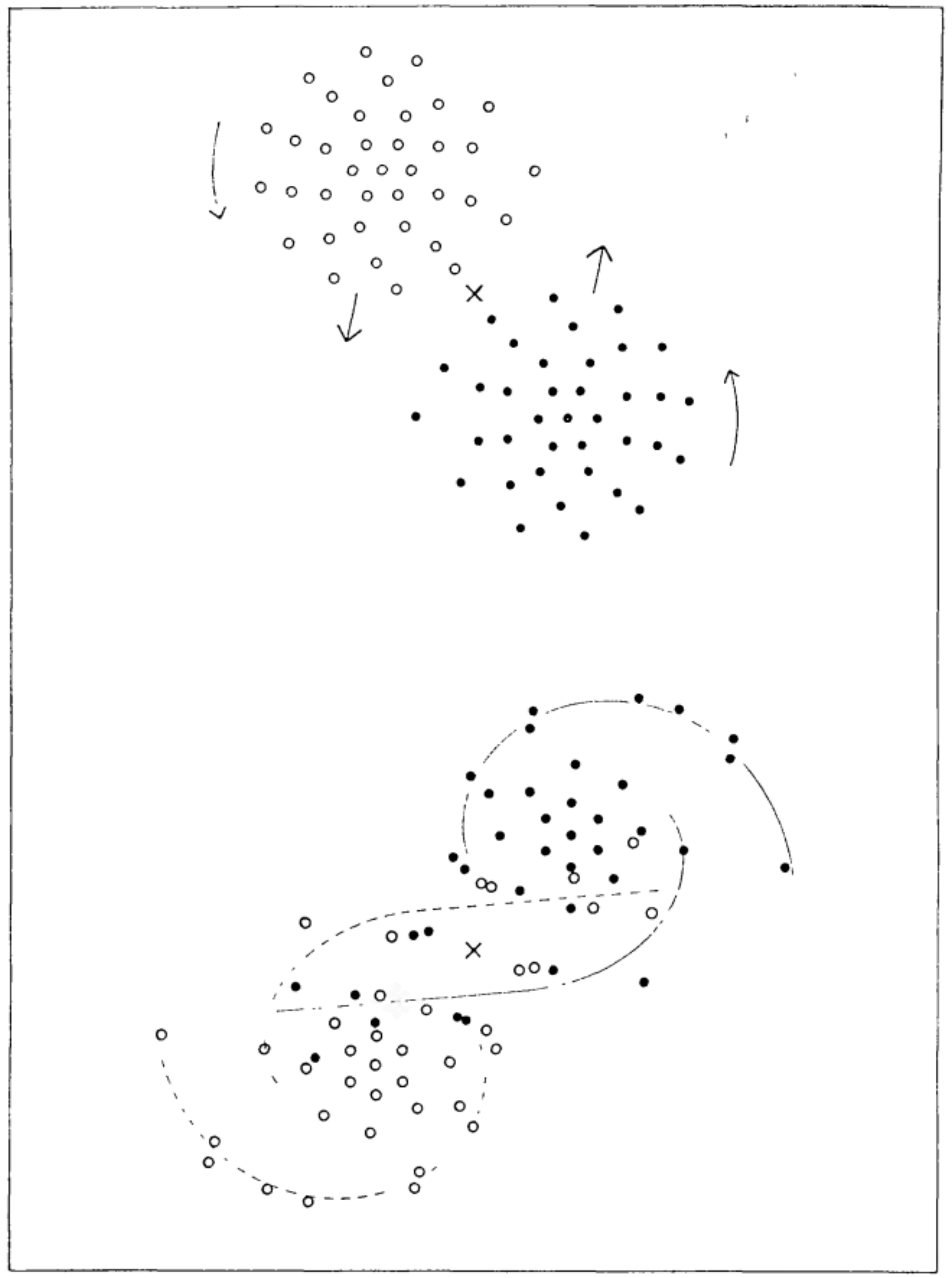

FIG. 4b

FIG. 4a.—Tidal deformations corresponding to parabolic motions, clockwise rotations, and a distance of closest approach equal to the diameters of the nebulae. The spiral arms point in the direction of the rotation.

FIG. 4b.—Same as above, with the exception of counterclockwise rotations. The spiral arms point in the direction opposite to the rotation.

Computing forces between particles: Softening

- The equation of motion is

$$\ddot{\mathbf{r}}_i = -G \sum_{j=1,\neq i}^{N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^3}$$

- Singularity as $\left|\mathbf{r}_i - \mathbf{r}_j\right| \rightarrow 0$ → can cause very small time steps
- Replace denominator with $\left(\left|\mathbf{r}_i - \mathbf{r}_j\right|^2 + \varepsilon^2\right)^{3/2}$

$$\mathbf{F}_i = \sum_{i\neq j} \frac{G m_i m_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^2 + \epsilon^2}$$

- ε is a small constant: softening parameter
- This approximation may be justifiable if close encounters between particles are unimportant (e.g., galaxy dynamics)
- Not necessarily good for modelling star clusters → physically eliminates formation of binaries with $r < \epsilon$
- Hard binaries (very small separation) are an important source of energy in clusters

Modelling globular cluster (collisional) stellar dynamics

- Particle methods:
  - **Direct summation *N*-body approach; “brute force”**
    - NBODYX series of codes: https://people.ast.cam.ac.uk/~sverre/web/pages/nbody.htm (Aarseth 2003)
    - NBODY6++GPU (Wang, Spurzem et al. 2015; 2016): https://github.com/nbody6ppgpu
  - **Monte Carlo method**
    - MOCCA (Giersz 1998 → Hypki & Giersz 2013) http://www.moccacode.net/
    - CMC (Joshi et al 2000 → Rodriguez et al. 2022) https://clustermontecarlo.github.io/CMC-COSMIC/
- Continuum methods/Phase space descriptions:
  - Gas sphere
  - Fokker-Planck methods

Also see: http://www.artcompsci.org/ & https://github.com/amusecode/amuse

Sverre Aarseth

Michel Hénon

Speed–Accuracy tradeoff

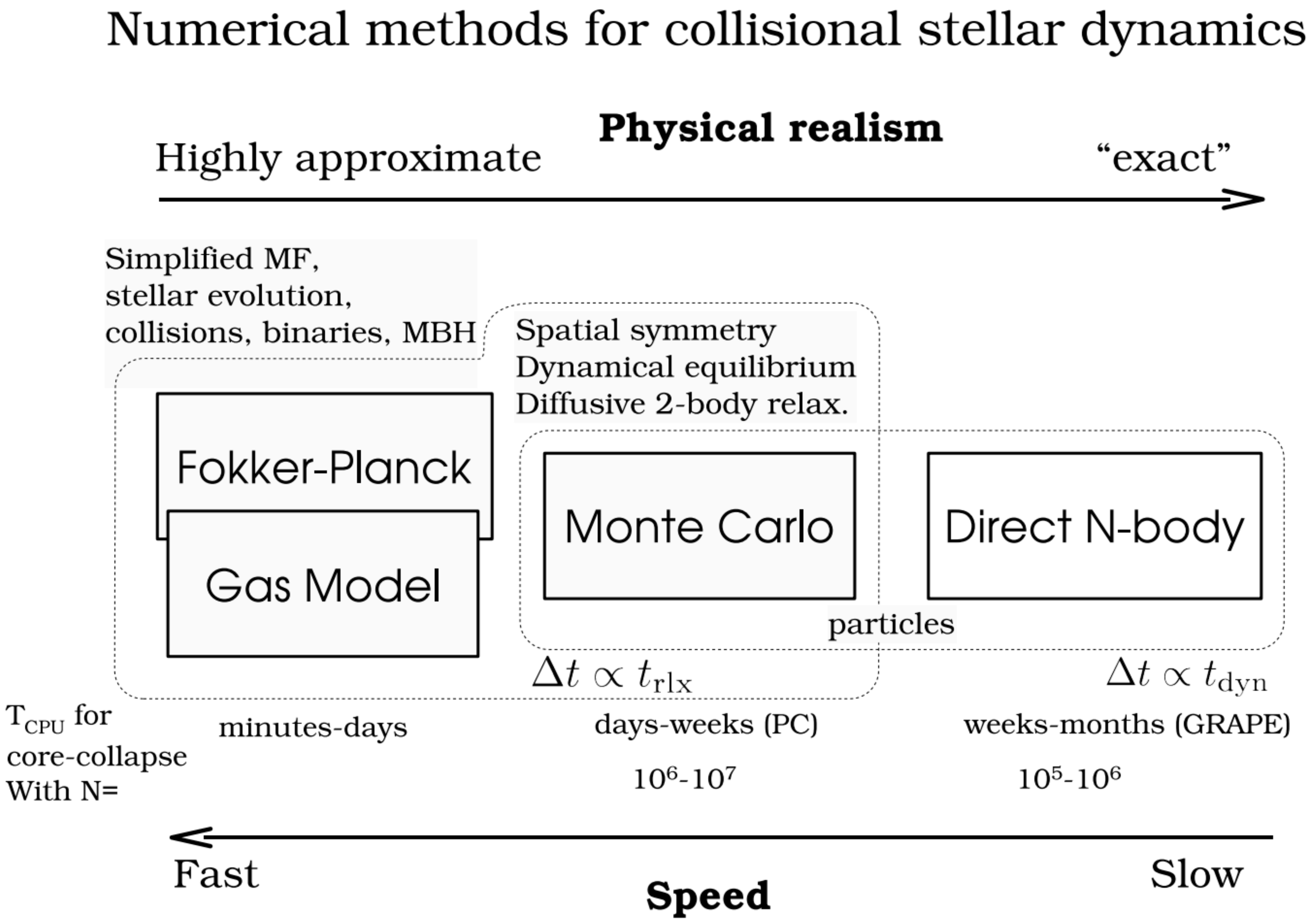

Fig 3 from Amaro-Seoane et al. 2007

Equilibrium models and initial conditions for cluster models

- Mass, 3 position coordinates, 3 velocity coordinates for N stars

$$f(\overrightarrow{x}, \overrightarrow{v}, t)d^3\overrightarrow{x}d^3\overrightarrow{v}$$

- Systems evolve towards dynamical equilibrium:

$$\mathbf{v}_i \equiv \dot{r}_i$$

Continuous sequence of changes in response to an independent variable, t.
We need to specify 6N conditions (x and v) to solve our initial value problem.

  - No overall expansion or contraction of the system, or other bulk motion, even though all particles are in motion → **"virial equilibrium"**

$$K = \sum_i \frac{1}{2} m_i v_i^2$$

$$U = -\sum_{i,j,i\neq j} \frac{G m_i m_j}{r_{ij}}$$

$$2K + U = 0$$

$$K + U = E$$

$$K = -E$$

$$U = 2E$$

$$M = \sum_{i=1}^{N} m_i$$

  - V - typical stellar speed
  - M - total cluster mass
  - R - virial radius

$$V^2 = \frac{2K}{M}$$

$$U = -2K$$

$$U = -\frac{GM^2}{2R}$$

$$Q = \frac{K}{|U|} = 0.5$$

N-body units (also known as Hénon units)

$$G = 1$$
$$M = 1$$
$$R = 1 \quad \text{(virial radius)}$$

- In these units:
  - The characteristic speed $$V^2 = \frac{GM}{2R} = \frac{1}{2}$$
  - The crossing time $$t_{cr} = \frac{2R}{V} = 2\sqrt{2}$$
  - The total energy $$E = -\frac{1}{2}MV^2 = -\frac{1}{4}$$

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Why should we use *N*-body units?

- choosing right units helps the integrator → rounding errors
- Scalability of results

N-body units (also known as Hénon units)

- This is a conventional system of units in which:

$$G = 1$$
$$M = 1$$
$$R = 1$$

- These are often used in N-body and Monte Carlo simulations.
- Example of scaling from N-body units:
  - Velocity of a system is given by $V^2 = \frac{GM}{2R}$
  - Star cluster with $M = 10^5 M_\odot$ and $R = 5 \text{ pc}$
  - To convert a velocity from N-body codes to km/s, multiply by $\sqrt{\frac{GM}{R}}$
  - where G is expressed in the same units as mass, radius and velocity (i.e. km/s, $M_\odot$, pc)

Reminder: What do we need to integrate?

The equations of motion to be integrated are

$$\ddot{\mathbf{r}}_i = -G \sum_{j=1,\neq i}^{N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

These can be written in the equivalent form

$$\dot{\mathbf{r}}_i = \mathbf{v}_i$$

$$\dot{\mathbf{v}}_i = \mathbf{a}_i = -\sum_{j=1,j\neq i}^{N} G m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

where $\mathbf{r}_i$, $\mathbf{v}_i$ are the position and velocity of the i th particle.

- To solve the ordinary differential equations (2 coupled ODEs)
- Accurate time integration of close encounters is the most difficult part of collisional *N*-body methods
- For collisionless N-body methods force softening alleviates this problem substantially.

Taylor series expansion of the equations of motion

$$x_i(t+\Delta t) = x_i(t) + \frac{\mathrm{d}x_i(t)}{\mathrm{d}t}\Delta t + \frac{1}{2}\frac{\mathrm{d}^2 x_i(t)}{\mathrm{d}t^2}\Delta t^2 + O\left(\Delta t^3\right)$$

$$v_i(t+\Delta t) = v_i(t) + \frac{\mathrm{d}v_i(t)}{\mathrm{d}t}\Delta t + \frac{1}{2}\frac{\mathrm{d}^2 v_i(t)}{\mathrm{d}t^2}\Delta t^2 + O\left(\Delta t^3\right)$$

- Obtaining position and velocities at time $t + \Delta t$ knowing info at time t

Integration strategy: Euler method

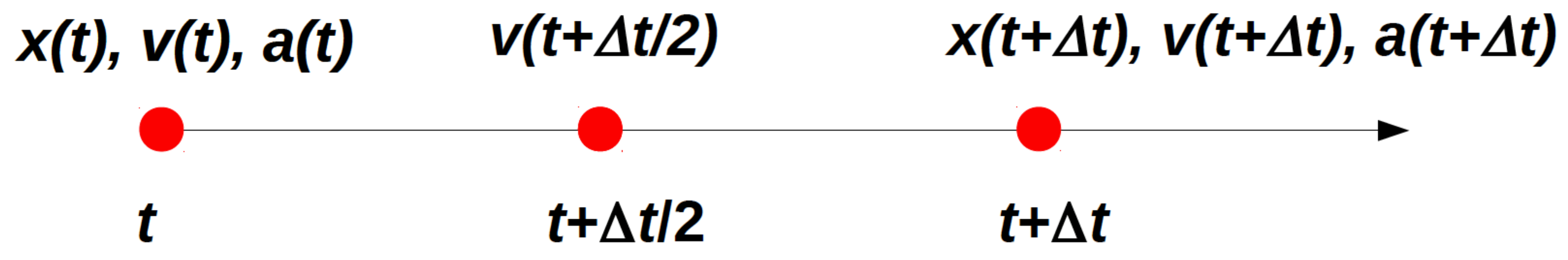
- Updates the position and velocity for a given particle by time step Δt via

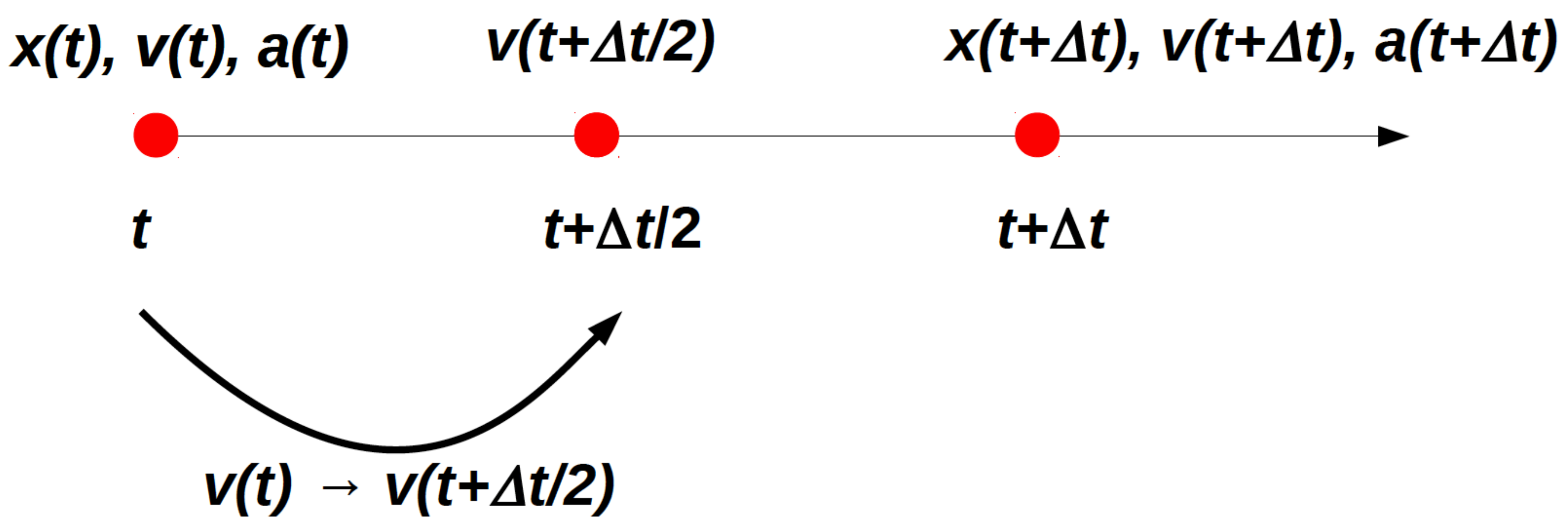
$$r(t + \Delta t) = r(t) + v(t)\Delta t$$
$$v(t + \Delta t) = v(t) + a(t)\Delta t$$

- In this approximation, the velocity and acceleration of the particle are held constant for the duration of the time step.
- While conceptually straightforward, this scheme performs very poorly in practice
  - The Euler method is just a Taylor expansion to first order in Δt and the errors are proportional to Δt^2
  - Errors grow to quickly to be used to study astrophysical systems like star clusters

Integration strategy: Leapfrog method

- Similar to Euler but evalutations are done in between time step Δt





Leapfrog algorithm

$$v\left(t+\frac{\Delta t}{2}\right) = v(t) + a(t)\frac{\Delta t}{2}$$

$$r(t+\Delta t) = r(t) + v(t)\Delta t + \frac{1}{2}a(t)\Delta t^2$$

$$v(t+\Delta t) = v\left(t+\frac{\Delta t}{2}\right) + a(t+\Delta t)\frac{\Delta t}{2}$$

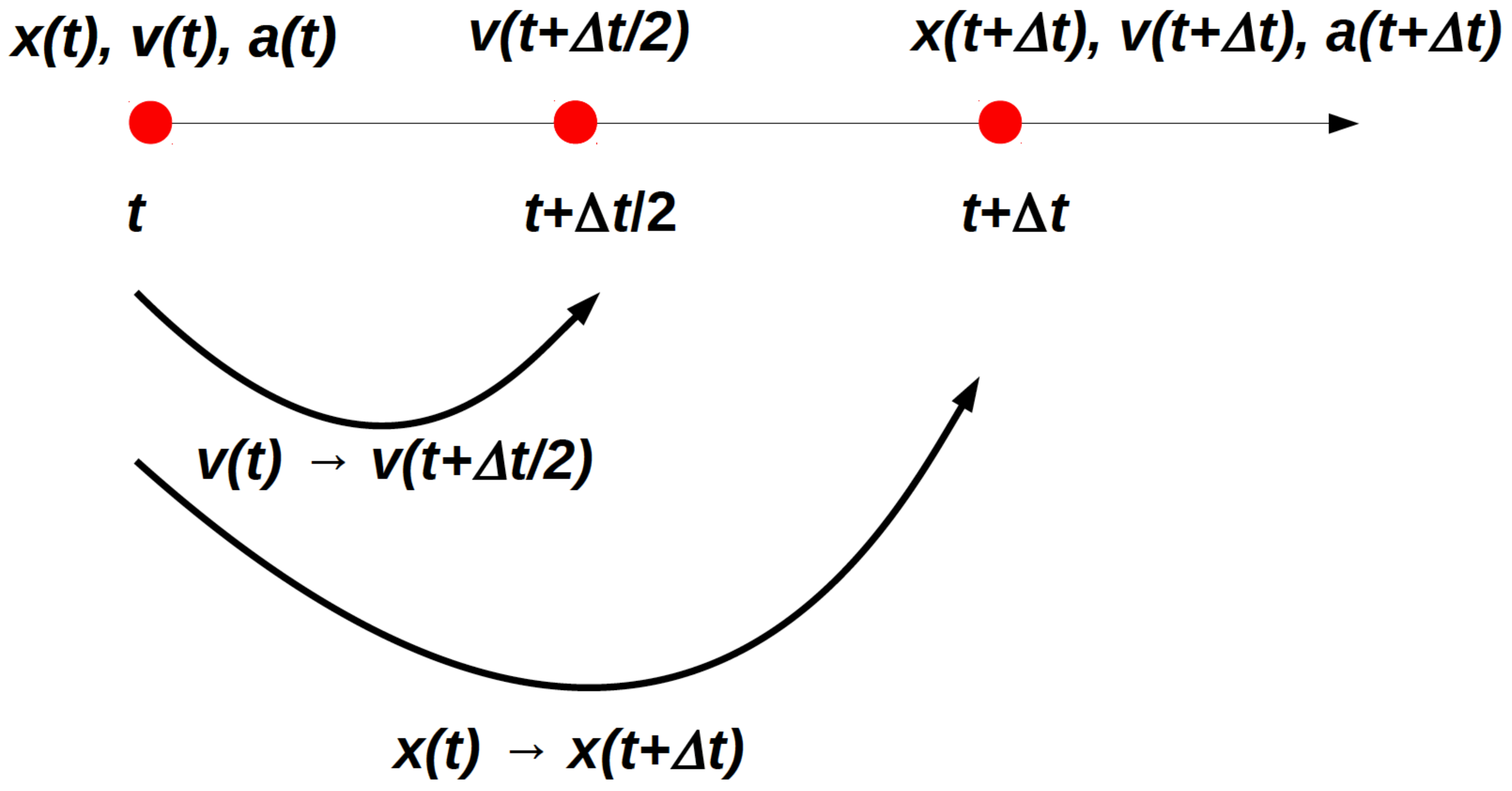
New position is calculated using an extra term proportional to Δt^2

Velocity updated in 2 steps - first half of the time step is taken using the current acceleration and second is taken using the new acceleration

Credit: Mapelli lectures on *N*-body techniques in astrophysics

Integration strategy: Leapfrog method

- Similar to Euler but evalutations are done in between time step Δt



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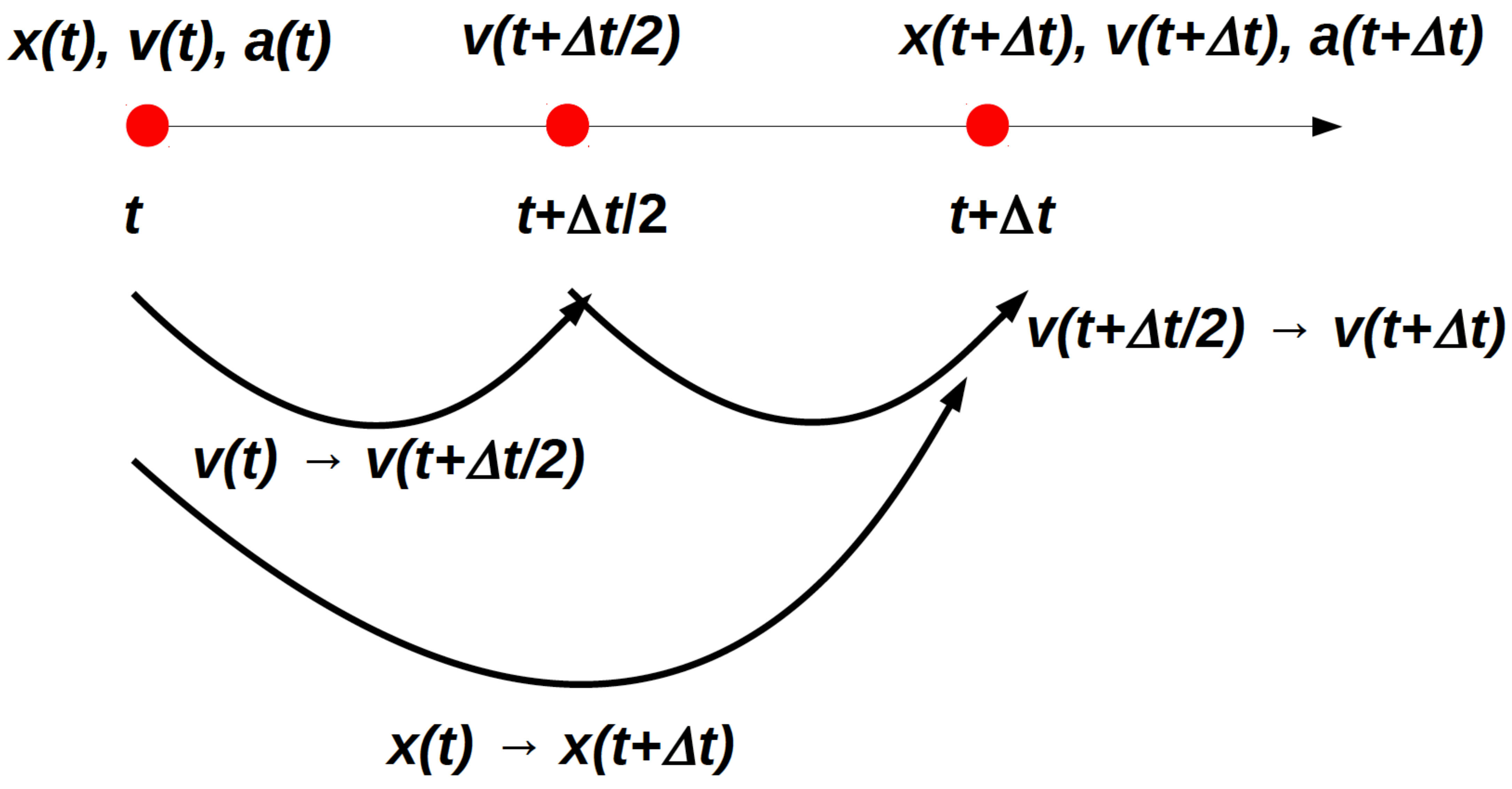

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Credit: Mapelli lectures on *N*-body techniques in astrophysics

Implementation of direct *N*-body codes for collisional dynamics

- **Direct summation *N*-body approach; "brute force"**
  - NBODYX series of codes: https://people.ast.cam.ac.uk/~sverre/web/pages/nbody.htm (Aarseth 2003)
  - NBODY6++GPU (Wang, Spurzem et al. 2015; 2016): https://github.com/nbody6ppgpu
- **Integration Scheme:**
  - Since close encounters and interactions between stars are important in star clusters → integrator must be high accuracy even on short times scales → 4th order accuracy
  - Expand Taylor series solution for the position and velocities to fourth order in an interval → Hermite integrator

$$\mathbf{r}_i(t+\Delta t) = \mathbf{r}_i(t) + \Delta t \mathbf{v}_i(t) + \frac{1}{2}(\Delta t)^2 \mathbf{a}_i(t) + \frac{1}{6}(\Delta t)^3 \mathbf{j}_i(t) + \dots$$

Euler method Algorithm:

$$r(t+\Delta t) = r(t) + v(t)\Delta t$$
$$v(t+\Delta t) = v(t) + a(t)\Delta t$$

Leapfrog algorithm

$$v\left(t+\frac{\Delta t}{2}\right) = v(t) + a(t)\frac{\Delta t}{2}$$

$$r(t+\Delta t) = r(t) + v(t)\Delta t + \frac{1}{2}a(t)\Delta t^2$$

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New position is calculated using an extra term proportional to Δt^2
Velocity updated in 2 steps - first half of the time step is taken using the current acceleration and second is taken using the new acceleration

Hermite Integration: 4th order predictor-corrector

- The algorithm consists of a prediction step:

$$r_p = r(t) + v(t)\Delta t + \frac{1}{2}a(t)\Delta t^2 + \frac{1}{6}j(t)\Delta t^3$$

$$v_p = v(t) + a(t)\Delta t + \frac{1}{2}j(t)\Delta t^2$$

Taylor series evaluation

Calculating the Jerk:

$$\mathbf{a}_i = -G\sum_{j=1,\neq i}^{N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^3}$$

$$\mathbf{j}_i \equiv \dot{\mathbf{a}}_i = -G\sum_{j=1,\neq i}^{N} m_j \left(\frac{\mathbf{v}_i - \mathbf{v}_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^3} - 3\frac{\left(\mathbf{v}_i - \mathbf{v}_j\right).\left(\mathbf{r}_i - \mathbf{r}_j\right)}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^5}\left(\mathbf{r}_i - \mathbf{r}_j\right) \right)$$

- a correction step that makes use of the initial coordinates and the predicted coordinates:

$$r(t+\Delta t) = r(t) + \frac{1}{2}\left(v(t) + v_p\right)\Delta t + \frac{1}{12}\left(a(t) - a_p\right)\Delta t^2$$

$$v(t+\Delta t) = v(t) + \frac{1}{2}\left(a(t) + a_p\right)\Delta t + \frac{1}{12}\left(j(t) - j_p\right)\Delta t^2$$

Hermite interpolation which approximates the higher accelerating terms by another Taylor series
Trick: Instead of doing more derivatives of jerk, use derivative of predicted values

- j(t) is the jerk which is the time derivative of the acceleration
- a_p is the acceleration calculated using the predicted positions

Hermite Integration: 4th order predictor-corrector

- The algorithm consists of a prediction step:

$$r_p = r(t) + v(t)\Delta t + \frac{1}{2}a(t)\Delta t^2 + \frac{1}{6}j(t)\Delta t^3$$

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$$\ldots - \mathbf{v}_j\big) \cdot \big(\mathbf{r}_i - \mathbf{r}_j\big) \over \left|\mathbf{r}_i - \mathbf{r}_j\right|^5} \big(\mathbf{r}_i - \mathbf{r}_j\big) \Bigg)$$

- a correction ste…

coordinates an…

$$r(t+\Delta t) = r(t) + \frac{1}{2}\Big(v(\ldots$$

$$v(t+\Delta t) = v(t) + \frac{1}{2}\Big(a(\ldots$$

4th order Hermite predictor-corrector scheme is 3 step:

1. **predictor step: predicts positions and velocities at 3rd order**
2. **calculation step: calculates acceleration and jerk for the predicted positions and velocities**
3. **corrector step: corrects positions and velocities using the acceleration and jerk calculated in 2**

…erpolation which …e higher accelerating …other Taylor series …doing more derivatives of jerk, use derivative of predicted values

- j(t) is the jerk which is the time derivative of the acceleration
- a_p is the acceleration calculated using the predicted positions

Basic structure of an *N*-body code

1. Initialisation of $\mathbf{r}_i, \mathbf{v}_i$, tnext $_i$ (update time $t_i + \Delta t_i$), $\mathbf{a}_i, \dot{\mathbf{a}}_i$ for all I

2. Choose i minimising tnext $_i$ ←

3. Extrapolate all $\boldsymbol{r}_j, \mathbf{v}_j$ to tnext $_{\boldsymbol{i}}$

4. Compute new $\mathbf{a}_i, \dot{\mathbf{a}}_i$ (Predictor step)

5. Correct new $\mathbf{r}_i, \mathbf{v}_i$ (Hermite integrator)

6. Compute new tnext $_i$ →

Note: This is the basic structure of NBODY6, except for the absence of block time steps

Basic structure of an *N*-body code and time step issues

1. Initialisation of $\mathbf{r}_i, \mathbf{v}_i$, tnext $_i$ (update time $t_i + \Delta t_i$), $\mathbf{a}_i, \dot{\mathbf{a}}_i$ for all I

2. Choose i minimising tnext $_i$

3. Extrapolate all $\boldsymbol{r}_j, \mathbf{v}_j$ to tnext $_i$

4. Compute new $\mathbf{a}_i, \dot{\mathbf{a}}_i$ (Predictor step)

5. Correct new $\mathbf{r}_i, \mathbf{v}_i$ (Hermite integrator)

6. Compute new tnext $_i$

$$\Delta t_i = \eta \frac{a_i}{j_i}$$

- Time step issues:
  - Same time step for all particles?
    - Expensive because a few particles undergo close encounters → force changes more rapidly for them
  - Ideally:
    - Longer time steps for 'unperturbed' particles
    - Shorter for particles that undergo close encounters
    - Different Δt_i for each particle is expensive and systems lose coherence
  - Block time step scheme: group particles by replacing their individual time steps such that $t/\Delta t_{i,b}$ is an integer (good for synchronization):
    Group together particles which have very similar update times. The extrapolation is shared among them.

Check out Aarseth, Tout & Mardling (eds): *The Cambridge N-Body Lectures (*2008) for details

Basic structure of an *N*-body code and time step issues

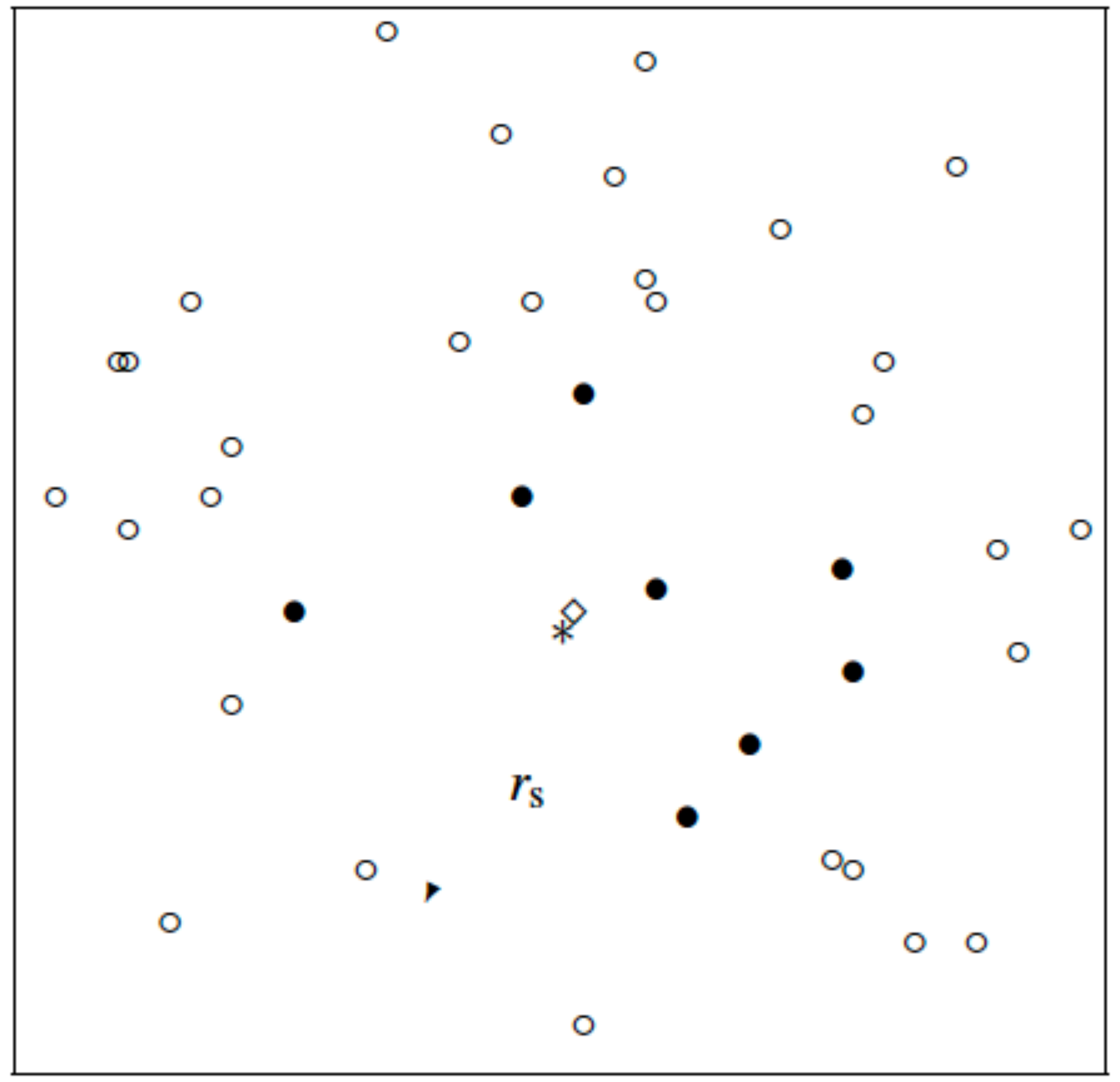


Figure 10.1: Illustration of the neighbour scheme for particle *i* marked as the asterisk (after [2]).

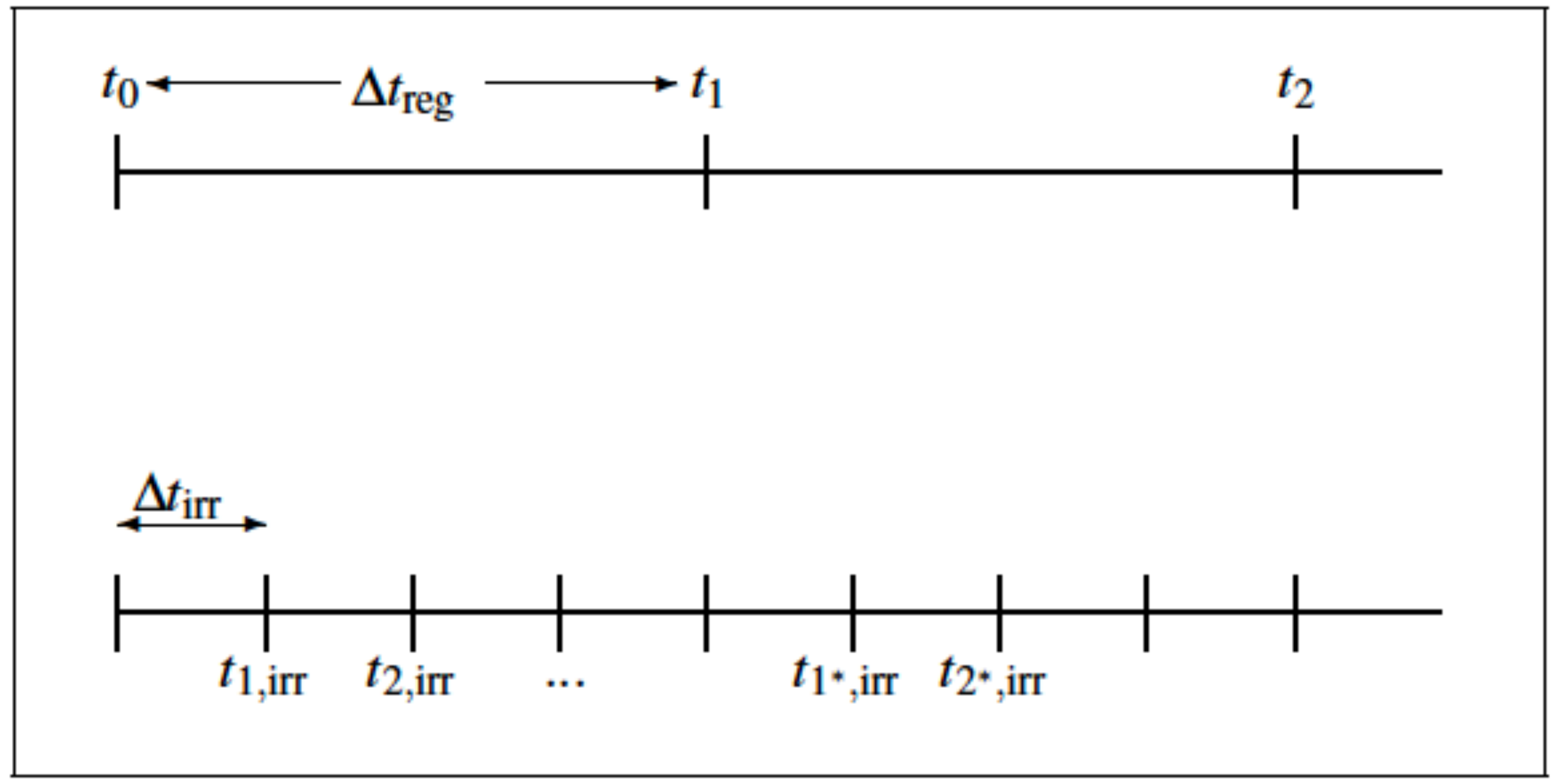


Figure 10.2: Regular and irregular time steps (after [22]).

- Neighbour Scheme (Ahmad & Cohen 1973)
  - Time step determined by nearest neighbour
  - Few near neighbours→ force due to them can be computed frequently with little effort (the "irregular force")
  - force due to the more numerous non-neighbours (the "regular force") fluctuates more slowly, and can be computed with a longer time step
  - Requires keeping a list of neighbours

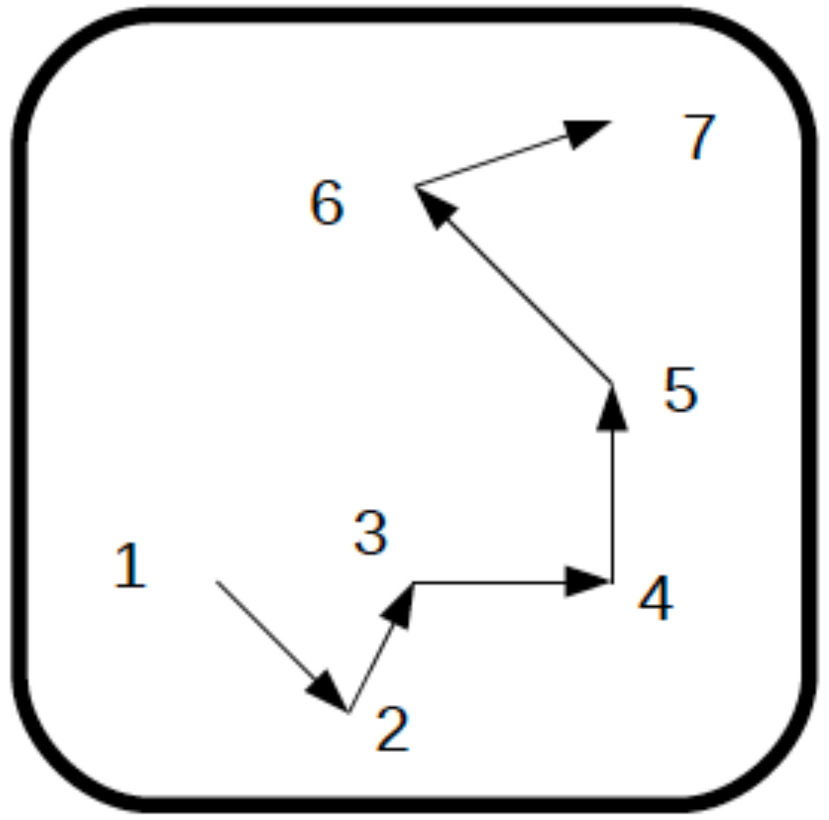
Check out Aarseth, Tout & Mardling (eds): *The Cambridge N-Body Lectures (*2008) for details + NBODY6++GPU Manual

Regularization: a way of handling close encounters

- Mathematical trick → remove the singularity in the Newtonian law of gravitation for two particles which approach each other arbitrarily close (change of variables, different from softening)
- Kustaanheimo-Stiefel (KS) regularization: for binaries and 3-body encounters
  - Change from coordinates to offset coordinates: CM and relative particle
  - Kepler orbit is transformed into a harmonic oscillator
    - Significantly reduces the number of steps needed to integrate the orbit and reduces round-off error
- CHAIN regularization (Mikkola & Aarseth 1993: for small N-body systems
  - Calculate distances between an active object (e.g. binary) and the closest neighbours
  - find vectors that minimize the distances → use these vectors ("chain coordinates")
  - to change coordinates and make suitable changes of time coordinates → calculate forces with new coordinates

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$x_{rel} = x_1 - x_2$$



Credit: Mapelli lectures on *N*-body techniques in astrophysics