Collisionless and Collisional Stellar Dynamics
Lecture II

Mirek Giersz and Abbas Askar

Nicolaus Copernicus Astronomical Center, Polish Academy of Sciences
Warsaw, Poland

mig@camk.edu.pl and askar@camk.edu.pl

Star Cluster Dynamics and Evolution

12 March 2024

variables written in bold are vectors
variables written in normal font are scalars

Let's consider a self-gravitating system of N stars that do not experience any external forces - the system is isolated. The stars are from 1 to N. Consider a Cartesian coordinate system x, y, z. The coordinates of star i are x_i, y_i, z_i. Accordingly the velocity has components

$$\begin{aligned} \frac{dx_i}{dt} &= \dot{x}_i = u_i \\ \frac{dy_i}{dt} &= \dot{y}_i = v_i \\ \frac{dz_i}{dt} &= \dot{z}_i = w_i \end{aligned} \tag{1}$$

The distance between stars i and j is

$$r_{ij} = |\mathbf{r_j} - \mathbf{r_i}| = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$$

The force of attraction between stars i and j is

$$\mathbf{F_{ij}} = \frac{G m_i m_j}{r_{ij}^2} \frac{\mathbf{r_j} - \mathbf{r_i}}{|\mathbf{r_j} - \mathbf{r_i}|} \tag{2}$$

where m_i and m_j are masses of star i and j, respectively.
The total force acting on star i due to all the other stars is

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F_i} = \sum_{j=1, i \neq j}^{N} \frac{G m_i m_j (\mathbf{r_j} - \mathbf{r_i})}{r_{ij}^3} \qquad i = 1, ..., N \tag{3}$$

We have a system of 3N simultaneous second-order differential equations - dynamical system of order 6N. The evolution of the system in time is know if we are able to solve 6N differential equations.

Center of Mass

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{N} \sum_{j=1, i\neq j}^{N} \frac{G m_i m_j (\mathbf{r_j} - \mathbf{r_i})}{r_{ij}^3}$$

$$\frac{G m_i m_j (\mathbf{r_j} - \mathbf{r_i})}{r_{ij}^3} + \frac{G m_j m_i (\mathbf{r_i} - \mathbf{r_j})}{r_{ji}^3} = 0$$

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i = 0$$

$$\sum_{i=1}^{N} m_i \mathbf{r_i} = \mathbf{a}t + \mathbf{b}$$

$$\frac{\sum_{i=1}^{N} m_i \mathbf{r_i}}{\sum_{i=1}^{N} m_i} = \frac{\mathbf{a}t + \mathbf{b}}{M} \qquad (4)$$

The center of mass of the system moves at constant speed

Total Angular Momentum

$$\sum_{i=1}^{N} m_i \mathbf{r_i} \times \ddot{\mathbf{r}}_i = \sum_{i=1}^{N} \sum_{j=1, i\neq j}^{N} \frac{G m_i m_j \mathbf{r_i} \times (\mathbf{r_j} - \mathbf{r_i})}{r_{ij}^3} \qquad i = 1, ..., N$$

Since term $\mathbf{r_i} \times \mathbf{r_i} = 0$ and $\mathbf{r_i} \times \mathbf{r_j} = -\mathbf{r_j} \times \mathbf{r_i}$ then

$$\sum_{i=1}^{N} m_i \mathbf{r_i} \times \ddot{\mathbf{r}}_i = 0$$

$$\frac{d}{dt}\left(\sum_{i=1}^{N} m_i \mathbf{r_i} \times \dot{\mathbf{r}}_i\right) = \sum_{i=1}^{N} m_i \mathbf{r_i} \times \ddot{\mathbf{r}}_i = 0$$

$$\sum_{i=1}^{N} m_i \mathbf{r_i} \times \dot{\mathbf{r}}_i = \mathbf{c} \tag{5}$$

The total angular momentum of the system is conserved for an isolated system

Total Energy

$$\Omega = -\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{Gm_im_j}{r_{ij}} \qquad \textit{Total Potential Energy}$$

$$\frac{\partial\Omega}{\partial x_i} = -\sum_{i=1,i\neq j}^{N}\frac{\partial}{\partial x_i}\left(\frac{Gm_im_j}{r_{ij}}\right) = \sum_{i=1,i\neq j}^{N}\frac{Gm_im_j}{r_{ij}^2}\frac{\partial r_{ij}}{\partial x_i} = -\sum_{i=1,i\neq j}^{N}\frac{Gm_im_j}{r_{ij}^2}\frac{x_j - x_i}{r_{ij}} = -m_i\ddot{x}_i$$

Finally

$$m_i\ddot{\mathbf{r}}_i = -\nabla_{\mathbf{i}}\Omega \tag{6}$$

The force equals minus the gradient of the potential energy

Scalar product of Eq.6 gives

$$\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i + \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \nabla_{\mathbf{i}} \Omega = 0$$

$$\frac{\partial \Omega}{\partial x_1}\dot{x}_1 + ... + \frac{\partial \Omega}{\partial z_N}\dot{z}_N = \frac{\partial \Omega}{\partial x_1}\frac{dx_1}{dt} + ... + \frac{\partial \Omega}{\partial z_N}\frac{dz_N}{dt} = \frac{d\Omega}{dt} \qquad \textit{Second Term}$$

$$\frac{d}{dt}\sum_{i=1}^{N} \frac{m_i \dot{\mathbf{r}}_i^2}{2} + \frac{d\Omega}{dt} = 0 \tag{7}$$

After integration over time Eq.7 we get

$$T + \Omega = E \tag{8}$$

The energy is constant

Concluding Remarks

- We have found ten integrals - the vectors **a**, **b** and **c** plus the scalar E
- In principle, each integral allows to substitute one variable in terms of the others, i.e., to reduce the order of the system by 1;
- the order of the system can be reduced from 6N to 6N - 10;
- Only the two-body problem (N = 2) is actually possible to solve completely;
- For real stellar systems, like star clusters, $N > 10^4$ only statistical approach can bring solutions - STATISTICAL MECHANICS;
- The integrals derived above refer to the whole system and not to individual stars;
- The symmetries of the systems allow the existence of such integrals, and they are very useful for constructing models of those systems.

$$I = \sum_{i=1}^{N} m_i \mathbf{r}_i^2 \qquad \textit{Moment of Inertia}$$

$$\dot{I} = 2 \sum_{i=1}^{N} m_i \mathbf{r}_i \dot{\mathbf{r}}_i$$

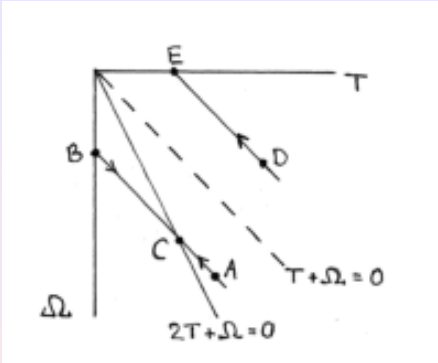
$$\ddot{I} = 2 \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i^2 + 2 \sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i$$

$$\ddot{I} = 4T + 2 \sum_{i=1}^{N} \sum_{j=1, i \neq j}^{N} \frac{G m_i m_j \mathbf{r_i} \times (\mathbf{r_j} - \mathbf{r_i})}{r_{ij}^3}$$

$$\mathbf{r}_i(\mathbf{r}_j - \mathbf{r}_i) + \mathbf{r}_j(\mathbf{r}_i - \mathbf{r}_j) = -(\mathbf{r}_j - \mathbf{r}_i)^2 = -r_{ij}^2$$

$$\ddot{I} = 4T - 2 \sum_{i=1}^{N} \sum_{j=1, i \neq j}^{N} \frac{G m_i m_j}{r_{ij}} = 4T + 2U$$

- Steady state, $\ddot{I} = 0$ - size of the system is constant
- Movement of stars on the crossing time scale makes $\ddot{I}$ not constant
- Averaging out statistical fluctuations gives the condition $2\overline{T} + \overline{\Omega} = 0$
- From the energy integral $T + \Omega = E$ and after time averaging we have $\overline{T} + \overline{\Omega} = E$
- So, $\overline{T} = -E$ and $\overline{\Omega} = 2E$, and $\overline{T}$ and $\overline{\Omega}$ are constant
- $T + \Omega = E$ so only lines with slope -1 are allowed
- Point A - $\ddot{I} > 0$, system expands to reduce T and Ω
- Point B - $\ddot{I} < 0$, system collapses to increase T and Ω



- This process is called **virialization**
- Poind D - system is dissolved
- **a necessary condition for stability of a stellar system is that** $E < 0$

Pratical Use of the Virial Theorem

The total kinetic energy

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 = \frac{1}{2} N \langle m v^2 \rangle$$

The total potential energy

$$\Omega = -\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{G m_i m_j}{r_{ij}} \simeq \frac{N(N-1)}{2} G \left\langle \frac{m_i m_j}{r_{ij}} \right\rangle$$

Assuming $\langle m v^2 \rangle = \langle m \rangle \langle v^2 \rangle$ and $\langle m_i m_j / r_{ij} \rangle = \langle m \rangle^2 / \langle r \rangle$ and $N\langle m \rangle = M$ we can write

$$\langle v^2 \rangle \simeq \frac{GM}{2\langle r \rangle} \tag{10}$$

Very important equation to estimate global properties of dynamical systems

- At a given moment t a star is fully described by its mass, m, position, (x, y, z), and velocity, (u, v, w). In principle, from this information one may compute the evolution of the system for all times.
- Each star, i, is represented by a point in 7-dimensional space called **phase space** - $x_i, y_i, z_i, u_i, v_i, w_i, m_i$.
- The whole system of N stars is represented by a set of N points in phase space. This set completely describes the state of the system.
- The N is very large. It is not possible to specify the exact position of each representative point. Instead we can specify only **the density of such points in phase space**.
- The phase space is divided into small cells. Each cell has to be small compared to the whole system but large enough to contain a considerable number of points.

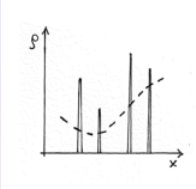
$$f = \frac{\Delta n}{\Delta \tau}$$

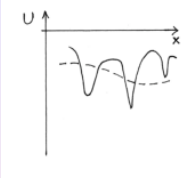
Where $\Delta\tau$ is the volume of the cell and Δn is the number of starts inside the cell.

- The value of f is defined for each cell. Interpolating and smoothing them between cells, a continuous function can be defined everywhere in phase space. This is an approximate function and approximation becomes better for larger N.
- f is a function of the seven phase space coordinates but also depends on time - representative points are moving as the stars follow their orbits.
- $f(x, y, z, u, v, w, m, t)$ is called distribution function (DF)

$$\int\int\int\int\int\int\int f dx dy dz du dv dw dm = \int f d\tau = N$$

Smoothed Density and Potential





$$\rho(x,y,z,t) = \int_o^m m dm \int\int\int_{-\infty}^{+\infty} f du dv dw \qquad (11)$$

$$U(x_1,y_1,z_1,t) = -G \int\int\int_{-\infty}^{+\infty} \frac{\rho(x_2,y_2,z_2,t) dx_2 dy_2 dz_2}{r_{12}}$$

$$4\pi G\rho = \nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \qquad (12)$$

ρ can be derived from DF, but the converse is not true

Poisson Equation

The Liouville Equation

$$\ddot{x} = -\frac{\partial U}{\partial x}; \quad \ddot{y} = -\frac{\partial U}{\partial y}; \quad \ddot{z} = -\frac{\partial U}{\partial z}$$

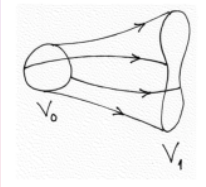
$$g = \begin{cases} \dot{x} & = u & = g_x \\ \dot{y} & = v & = g_y \\ \dot{z} & = w & = g_z \\ \dot{u} & = -\frac{\partial U}{\partial x} & = g_u \\ \dot{v} & = -\frac{\partial U}{\partial y} & = g_v \\ \dot{w} & = -\frac{\partial U}{\partial z} & = g_w \\ \dot{m} & = 0 & = g_m \end{cases}$$

- Seven equations describe the motion of a representative point in phase space. The velocity of the point is a vector g with components $(g_x, g_y, g_z, g_u, g_v, g_u, g_m)$.
- The curve described by a representative point in phase space is trajectory
- The orbit is a curve in physical space followed by the actual star.

The vector $\mathbf{g}$ has an important property

$$div\ \mathbf{g} = \nabla \cdot \mathbf{g} = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} + \frac{\partial g_u}{\partial u} + \frac{\partial g_v}{\partial v} + \frac{\partial g_w}{\partial w} + \frac{\partial g_m}{\partial m} = 0 \tag{13}$$

According to the definition $g_x = u$, so $\partial g_x/\partial x = 0$. Note: u and x are independent coordinates in phase space. The same is true for the two following terms. $g_u = \partial U/\partial x$ depends only on x, y, z, so $\partial g_u/\partial u = 0$. The same is true for the two following terms. $g_m = 0$



The value of f is constant along phase space trajectories

- Points at t_0 in volume V_0
- Points at t_1 in volume V_1
- If the flow of points is divergence free, $div\mathbf{g} = 0$ then $V_1 = V_0$
- **The fluid in the phase space is incompressible. Phase space volumes are conserved during the motion of the system**

The above arguments can be formulated by the principle of mass conservation. It can be written for any fuid

$$\frac{\partial f}{\partial t} + div\,(\mathbf{g}f) = 0$$

$$\frac{\partial f}{\partial t} + \mathbf{g}\cdot\nabla f + f\ div\ \mathbf{g} = 0$$

The third term vanishes and the scalar product gives

$$\frac{\partial f}{\partial t} + g_x\frac{\partial f}{\partial x} + ... + g_m\frac{\partial f}{\partial m} = 0$$

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} - \frac{\partial U}{\partial x}\frac{\partial f}{\partial u} - \frac{\partial U}{\partial y}\frac{\partial f}{\partial v} - \frac{\partial U}{\partial z}\frac{\partial f}{\partial w} = 0 \qquad \textbf{Boltzman Equation} \quad (14)$$

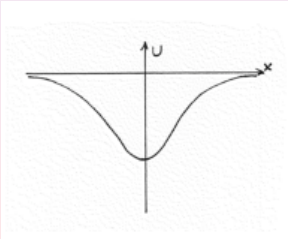
This equation expresses mathematically the fact that f is constant when following the motion

Phase Space Mixing

If the system is not in virial equilibrium, how long does it take for the distribution function to approach the steady state? Stars traversing the system exchange potential and kinetic energies on orbital time scale

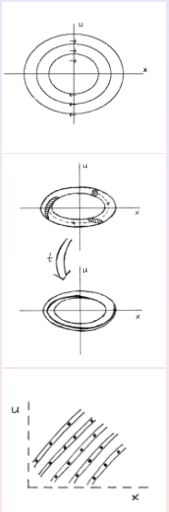
$$t_c = \frac{\langle r \rangle}{\langle v \rangle} = \sqrt{\frac{2\langle r \rangle^3}{GM}}$$

Crossing Time



Pendulum system. Motion occur in only one dimension. The phase space is then 2-dimensional - x and u

The potential U corresponds to an attractive force toward the origin. The objects of this system will therefore oscillate around the origin with different amplitudes,periods and phases.



- The trajectory is an oval curve in the (x, u) plane
- All pendulums initially cover a small area in a allowed phase space
- The area gets deformed due to the differences in period between pendulums. Note - in spite of this deformation, the area remains the same
- Eventually the area will look like a narrow, winding spiral
- As this spiral continues to wind, the different rounds get squeezed and squeezed. After some time it becomes impossible to separate the rounds from each other.
- For all practical purposes, the distribution of stars looks uniform within the band defined by the initial conditions. The distribution function is constant along the trajectories and does not change with time. This is the stationary state

Violent Relaxation

Violent relaxation changes the energies of the stars. When a star moves in a fixed potential U, its energy $E = 1/2v^2 + U$ is constant. But if U is a function $U(x,t)$ of space and time, E is not constant. $\frac{dE}{dt} = \frac{\partial U}{\partial t}|_{x(t)}$

Examples

(1) - Star at rest at the center of a collapsing spherical protogalaxy. As the protogalaxy collapses, the potential well at its center becomes deeper. On the other hand, the velocity of the central star remains zero. Therefore the energy of this star decreases.

(2) - Collapsing protoglaxy and slowly moving star outside half-mass radius. When the star will start to fall to the center the system will have the deepest central potential. The star at the center will acquire a lot of kinetic energy. Star moving again outside will file less deeper potential of re-expanding system. So, it will reach initial potential with much larger kinetic energy

Violent Relaxation widens the range of energies of the stars and is independent of the star's mass. Phase Mixing keeps energy constant.

The collisionless relaxation processes, Phase Mixing and Violent Relaxation are distinct effects but drive each other.

Jeans Theorem

Let's eliminate stellar mass from the distribution function, so we can write:

$$\varphi = \int_0^\infty f m dm$$

φ physically means the mass density in a 6-dimensional reduced phase space

$$\rho = \int\int\int_{-\infty}^{\infty} \varphi du dv dw$$

The Poisson equation remains unchanged

$$\frac{\partial \varphi}{\partial t} + u\frac{\partial \varphi}{\partial x} + v\frac{\partial \varphi}{\partial y} + w\frac{\partial \varphi}{\partial z} - \frac{\partial U}{\partial x}\frac{\partial \varphi}{\partial u} - \frac{\partial U}{\partial y}\frac{\partial \varphi}{\partial v} - \frac{\partial U}{\partial z}\frac{\partial \varphi}{\partial w} = 0 \qquad \textbf{Boltzman Equation} \quad (15)$$

By eliminating the masses in a collision-free system it does not matter at all, how the mass is distributed between different stars, as long as the total mass per unit volume is conserved. THIS IS NOT TRUE FOR COLLISIONAL SYSTEMS

Equation of motion are following

$$\frac{d\varphi}{dx} = u, ..., \frac{du}{dt} = -\frac{\partial U}{\partial x}, ... \qquad \textbf{six order differential equations}$$

An integral of the system is a function $I(x, y, z, u, v, w, t)$, which has a constant value, when inserting an arbitrary solution $x(t), ..., w(t)$ of the system

There is SIX independent integrals, $I_1, I_2, I_3, I_4, I_5, I_6$

every integral is a function of the six independent integrals

$$I = f(I_1, I_2, I_3, I_4, I_5, I_6) \tag{16}$$

This is the general expression for all integrals of the system, if f is an arbitrary function. Only six independent integrals are needed to know them all.

Differentiating I we get:

$$\frac{\partial I}{\partial x}u + ... - \frac{\partial I}{\partial u}\frac{\partial U}{\partial x} - ... + \frac{\partial I}{\partial t} = 0$$

This is the Boltzman equation, if we replace I by φ

The distribution function is an integral of the equations of motion

$$\varphi = f(I_1, I_2, I_3, I_4, I_5, I_6) \qquad \textbf{Jeans Theorem} \qquad (17)$$

The above expressions for I are not constrained to be explicitly independent of time. In the stationary state (system in equilibrium) $\partial\varphi/\partial t = 0$. Remember that ρ and U are independent on time in the steady state. **Therefore, we can choose the integrals of motions in such a way that only one of them will explicitly depend on time.**

Eliminating dt from equation of motion we get

$$\frac{dx}{u} = \frac{dx}{v} = \frac{dx}{w} = \frac{du}{-\frac{\partial U}{\partial x}} = \frac{dv}{-\frac{\partial U}{\partial y}} = \frac{dw}{-\frac{\partial U}{\partial z}}$$

U does not depend on time so this system of equations is independent on time. These are equation of trajectory

Let's chose one variable. x as an independant variable, all others will be dependent

$$\frac{dy}{dx} = \frac{v}{u}; \frac{dz}{dx} = \frac{w}{u}; ... \qquad \textbf{This is fifth order}$$

Hence there are five independent integrals, $I_i(x, y, z.u, v, w)$ that remain constant along any trajectory. They are integrals of the equations of motion

Among the six integrals,there are $I_1, ...I_5$ that are explicitly independent of time and are called conservative integrals, and one integral I_6 depends on time and is called a non-conservative integral

$$\varphi = \mathbf{f(I_1, I_2, I_3, I_4, I_5)} \qquad \textbf{For Steady State Systems } \varphi \textbf{ does not depend on time} \qquad (18)$$

non-isolating integral - the hyperplane consists of an infinite set of sheets situated infinitely close to each other – φ only isolating integrals

Short Summary

- Integrals of Motion: any function of the phase-space coordinates and time $I(\mathbf{x}, \mathbf{v}, t)$ that is constant along every orbit where $\mathbf{x}(t)$ and $\mathbf{v}(t)$ are a solution to the equations of motion
- Any function of integrals is also an integral
- Every integral is a constant of motion, but every constant of motion is not an integral. For a circular orbit in a spherical potential, the azimuthal speed $\psi = \Omega t + \psi_0$. Hence, $C(\psi, t) \equiv t - \psi/\Omega$ will be constant of motion, but is not an integral of motion because it depends on time
- Isolating Integrals of Motion reduce the dimensionality of the orbit by one, e.g E and J
- Non-Isolating Integrals of Motion do not affect the phase-space distribution of an orbit, i.e. do not reduce the dimensionality of the motion
- For steady state distribution function is only function of conservative and isolating integrals

Simple examples for spherically symmetric and isotropic stellar system - Polytropic models Distribution Function only function of energy. Assume that $E \to U - 1/2V^2$, where $U \to -U + U_0$

$$f(E) = \begin{cases} FE^{n-\frac{3}{2}} & E \geq 0 \\ 0 & E < 0 \end{cases}$$

Solution for density is:

$$\rho = c_n U^n \tag{19}$$

where c_n is the integration constant For the polytropic equation $\rho = Kp^\gamma$, where p is pressure, and finally

$$\rho^{\gamma-1} = \frac{\gamma - 1}{\gamma K} U \tag{20}$$

So $\gamma = 1 + 1/n$. For $n = 5$ there is the Plummer model and for $n = \infty$ there is the isothermal model

Plummer model - n = 5

$$\rho(r) = \frac{3M}{4\pi a^3}\left(1+\frac{r^2}{a^2}\right)^{-\frac{5}{2}}$$
$$M(r) = M\left(1+\frac{r^2}{a^2}\right)^{-\frac{3}{2}}$$
$$U(r) = -\frac{GM}{a}\left(1+\frac{r^2}{a^2}\right)^{-\frac{1}{2}} \quad (21)$$

where a is related to the total Potential Energy $W = -\frac{3\pi}{32}\frac{GM^2}{a}$

Isothermal Model - $n = \infty$ and $\gamma = 1$ Let's assume the following distribution function

$$f = f_0 exp\left(\frac{E}{\sigma^2}\right)$$

The distribution of velocities at any point is Maxwellian, with one-dimensional dispersion σ^2. The density $\rho = m \int f d\mathbf{v}$ at any point can be written as

$$\frac{\rho}{\rho_0} = exp\left(\frac{-(U - U_0)}{\sigma^2}\right)$$

where $kT/m = \sigma^2$, and ρ_0 and U_0 are the density and potential (respectively) at some point, often taken to be the center

We see that the density of more massive stars decreases with increasing U more quickly than the density of less massive stars. This behavior is a simple example of mass segregation. Note that it depends on the implicit assumption that all stars have the same temperature.

Poisson's equation

$$\frac{d}{dt}\left(r^2\frac{dln(\rho)}{dr}\right) = -\frac{4\pi G}{\sigma^2}r^2\rho$$

Solution - infinite central density and infinite total mass

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}, \qquad M(r) = \frac{2\sigma^2 r}{G}$$

To remove singularity in the center let's define new variables

$$\rho \to \frac{\rho}{\rho_0}, \qquad r \to \frac{r}{r_0}, \qquad r_0 = \left(\frac{9\sigma^2}{4\pi G\rho_0}\right)^{\frac{1}{2}}$$

Tho parameters: σ and ρ_0

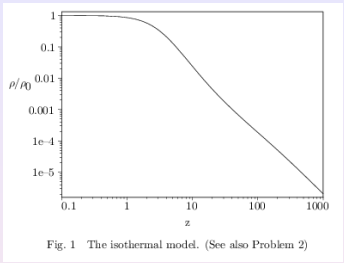


Fig. 1 The isothermal model. (See also Problem 2)

Useful approximation

$$\frac{\rho}{\rho_o} = 2\left(\frac{r}{r_0}\right)^{-2} \qquad \textbf{For} \quad r \gg r_0$$

$$\frac{\rho}{\rho_o} = \frac{1}{\left(1+(\frac{r}{r_0})^2\right)^{\frac{3}{2}}} \qquad \textbf{For} \quad r < r_0$$

Lowered Isothermal Model - King Model

A model that resembles the isothermal sphere at small radii, but its total mass is finite. We can do it by setting arbitrary constant U_0 such that the critical value of $E_0 = 0$

$$f(E) = \begin{cases} \rho_o (2\pi\sigma^2)^{-\frac{3}{2}} \left(e^{\frac{E}{\sigma^2}} - 1 \right) & E > 0 \\ 0 & E \leq 0 \end{cases}$$

$$U(r_t) = -\frac{GM(r_t)}{r_t}, \qquad r_t \ \ is \ \ the \ \ tidal \ \ radius$$

Additional model parameter - $U_0/\sigma^2 = W_0$
Concentration parameter $c = log_{10}(r_t/r_c)$

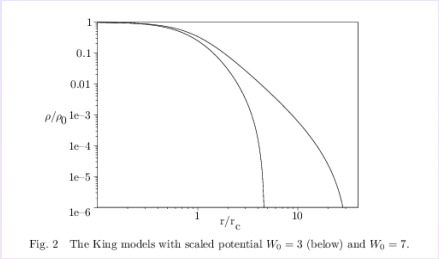


Fig. 2 The King models with scaled potential $W_0 = 3$ (below) and $W_0 = 7$.

The central potential is then $U(0) = U(r_t) - U_0$. The bigger the starting integration value U_0 the greater will be the tidal radius, the total mass, and $U(0)$

King models form a sequence that may be parameterized in terms of either c or W_0

Each star is defined by its position $\mathbf{r} = (x, y, z)$, its velocity $\mathbf{V} = (u, v, w)$, and its mass m. It is represented by a point in seven-dimensional space (x, y, z, u, v, w, m). A system of N stars is represented by N points in the phase space. The distribution function $f(\mathbf{r}, \mathbf{V}, m, t)$ is defined as a number density of these points in the phase space.

Mass density ρ and potential U are smoothed-out functions.

$$\rho(\mathbf{r}, t) = \int\int f d\mathbf{V} m dm$$

$$U(\mathbf{r}, t) = -G \int \frac{\rho(\mathbf{r}^{'}, t) d\mathbf{r}^{'}}{\mid \mathbf{r}^{'} - \mathbf{r} \mid}$$

$$\mathbf{a} = -\frac{\partial U}{\partial \mathbf{r}}$$

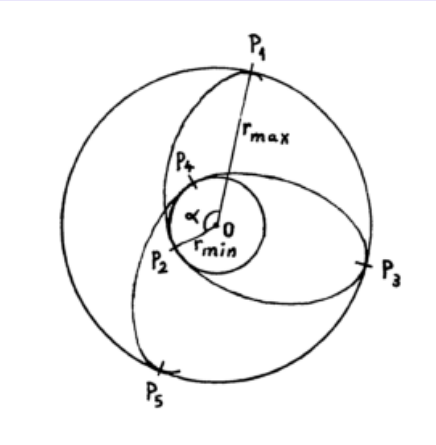
$$4\pi G\rho = \bigtriangledown^2 U \qquad \textit{Poisson's Equation}$$

The evolution of the distribution function is governed by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{V}\frac{\partial f}{\partial \mathbf{r}} + \mathbf{a}\frac{\partial f}{\partial \mathbf{V}} = 0$$

For the steady state the distribution function does not explicitly depends on time, so it is not changing in time - $\frac{\partial f}{\partial t} = 0$. The distribution function according to the Jeans theorem depends only on the **FOUR CONSERVATIVE, ISOLATING INTEGRALS OF MOTIONS**, $f(I_1, I_2, I_3, I_4, I_5, I_6)$. The six integral of motion is non-conservative and the five is non-isolating. Non-isolating integrals are such that the subspace defined by $I = const$ in phase space is made of an infinity of densely packed sheets.

The distribution function is only function of energy E, angular momentum $\mathbf{A}$ and mass, m – $f(E, \mathbf{A}, m)$. For spherically symmetric system dependence on angular momentum is only through its modulus.



For spherically symmetric system $\rho(r)$, $U(r)$ are only functions distance to the center, r. A star orbit is a rosette with peri and apo distances, r_{min} and r_{max}, respectively. The angle α between the consecutive min and max positions is constant. It is not commensurable with 2π - non isolating integral.

We can estimate the total kinetic and potential energies by $T = 0.5MV^2$, and $W = -GM^2/2R$. According to the virial theorem $2T + W = 0$, so the average velocity and the crossing time are:

$$V^2 = \frac{GM}{2R}$$

$$t_c = \left(\frac{2R^3}{GM}\right)^{\frac{1}{2}}$$