GALAXIES
Lecture 3-4

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Plan

1. Structure of galaxies: haloes and disks, circular velocities, NFW profile, exponential disk, Sersic profile
2. The Milky Way and galaxies of the Local Group
3. Orbits of stars in different potentials
4. Distribution functions, Jeans modeling, orbit-superposition models
5. Bars in galaxies: formation, evolution, orbital structure, dependence on environment
6. Spiral structure: geometry of spiral arms, formation scenarios
7. Interactions: tidal evolution and mergers, properties of merger remnants
8. Galaxy formation in cosmological context, cold and hot dark matter scenarios, top hat model, problems of theory on small scales

Types of orbits

- What kind of orbits are possible in galactic systems with different degrees of symmetry?
- Analytic results can be obtained for simpler potentials
- In general orbits become more complicated with decreasing symmetry of the potential
- We will assume that the gravitational fields of galaxies are smooth, neglecting perturbations due to stars, globular clusters or molecular clouds

Spherical potentials

- In a spherical potential the acceleration is directed towards the center of the mass distribution, therefore

$$\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)=\frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt}+\vec{r}\times\frac{d^2\vec{r}}{dt^2}=0$$

- Thus the angular momentum is conserved

$$\vec{r}\times\frac{d\vec{r}}{dt}=\vec{L}=const$$

- And the star moves in a plane

Equations of motion

- Since the motion is in the plane, we can use planar cylindrical coordinates (r, ψ) to describe it
- Equations of motion are given by

$$\ddot{r} - r\dot{\psi}^2 + \frac{d\Phi}{dr} = 0$$

$$\frac{d}{dt}\left(r^2\dot{\psi}\right) = 0$$

- The second equation is just the angular momentum conservation L=const

The energy

- The energy of the orbit in a spherical potential is

$$\frac{\dot{r}^2}{2} + \frac{\left(r\dot{\psi}\right)^2}{2} + \Phi(r) = E \qquad \text{or}$$

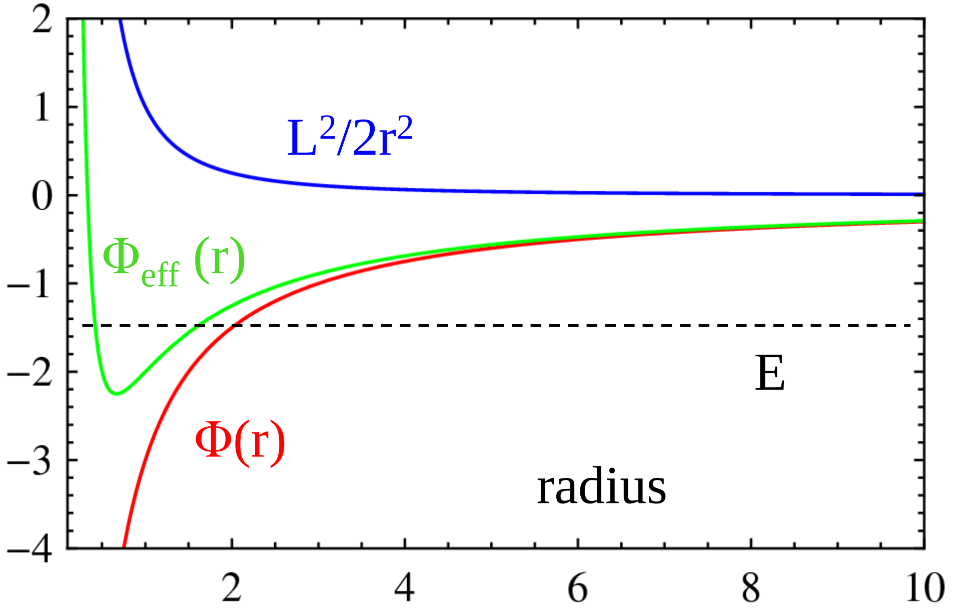
$$\frac{\dot{r}^2}{2} = E - \left[\Phi(r) + \frac{L^2}{2r^2}\right] = E - \Phi_{eff}(r)$$

- The star will stop when *dr*/*dt*=0 and this equation usually has two solutions: the apocenter and the pericenter distance

The effective potential

- The star moves as if in an effective potential

$$\Phi_{eff}(r) = \Phi(r) + \frac{L^2}{2r^2}$$



- The motion is between two radii defined by

$$E = \Phi_{eff}(r)$$

- The circular orbit corresponds to the minimum of the effective potential

Types of orbits

- Solutions of the equations of motion can be in the form of unbound orbits (r can be infinite)
- or bound orbits: r oscillates between finite limits, pericenter and apocenter
- The difference between the values of apo- and pericenter is the measure of eccentricity
- More eccentric orbits are called radial, more circular orbits are called tangential
- Distribution of orbits is important in modelling of galaxies

Periods

- The radial period of the orbit is the time required for the star to travel from apocenter to pericenter and back

$$T_r = 2\int_{r_p}^{r_a} \frac{dr}{\sqrt{2[E-\Phi(r)] - L^2/r^2}}$$

- The azimuthal period is the time required to cover the full angle of 2π

$$T_\psi = \frac{2\pi}{|\Delta\psi|} T_r$$

- Δψ is the angle covered during the radial period

$$\Delta\psi = 2\int_{r_p}^{r_a} \frac{d\psi}{dr} dr = 2\int_{r_p}^{r_a} \frac{L}{r^2}\frac{dt}{dr} dr = 2L\int_{r_p}^{r_a} \frac{dr}{r^2\sqrt{2[E-\Phi(r)] - L^2/r^2}}$$

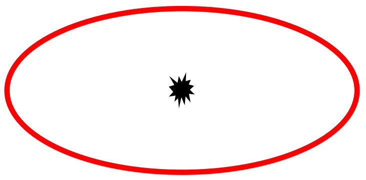
Closed orbits

- There are only two potentials in which all bound orbits are closed i.e. $\Delta\psi/2\pi$ is a rational number

- Spherical harmonic oscillator $\Phi(r) = \Omega^2 r^2/2 + const$

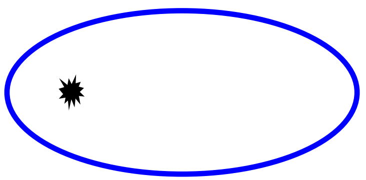
  (generated by a uniform sphere of matter)

$$T_\psi = 2\pi/\Omega, \quad T_r = T_\psi/2$$



- Kepler potential $\Phi(r) = -GM/r$

$$T_r = T_\psi$$



Orbits in galaxies

- Orbits of stars in a typical (spherical) galaxy can be understood as an intermediate case between the two types of closed orbits
- Typical galactic potentials are more extended than point masses and less extended than homogeneous spheres
- Since for a Keplerian orbit the radial period corresponds to $\Delta\psi=2\pi$ and for homogeneous sphere to $\Delta\psi=\pi$, in real galaxies

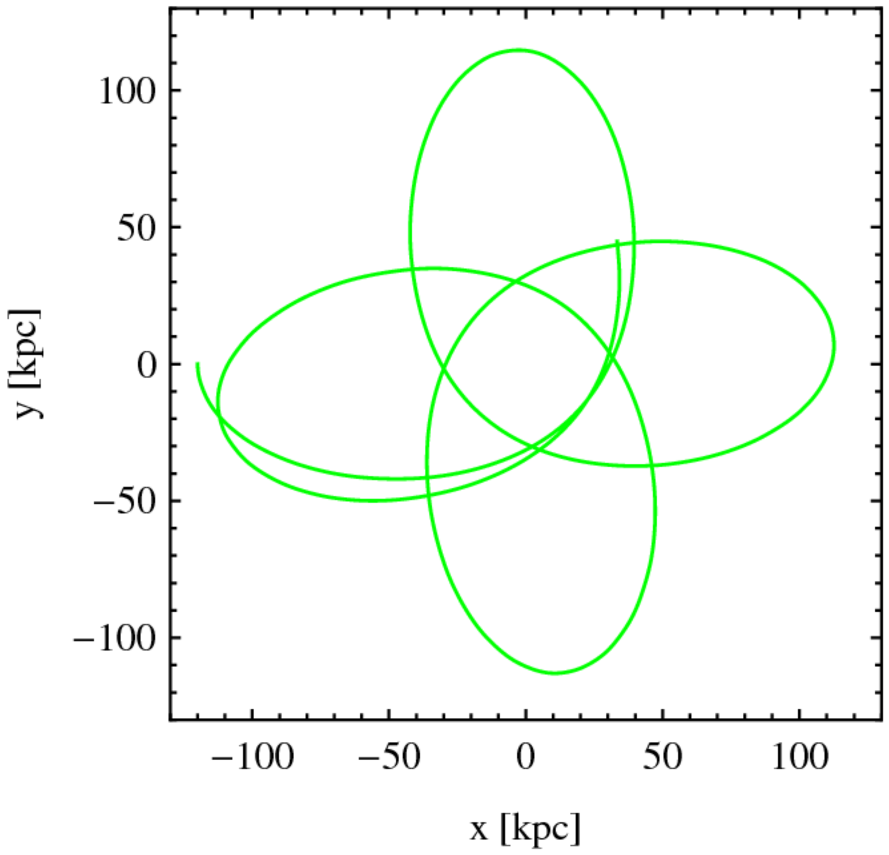
$$\pi < \Delta\psi < 2\pi$$

Angular momentum

- Radial orbits have low angular momentum so $\Delta\psi \rightarrow \pi$ for $L \rightarrow 0$
- Circular orbits have large angular momentum so

  $\Delta\psi \rightarrow 2\pi$ for $L \rightarrow L_{max}$
- This result is quite general, can be problematic only for some singular potentials

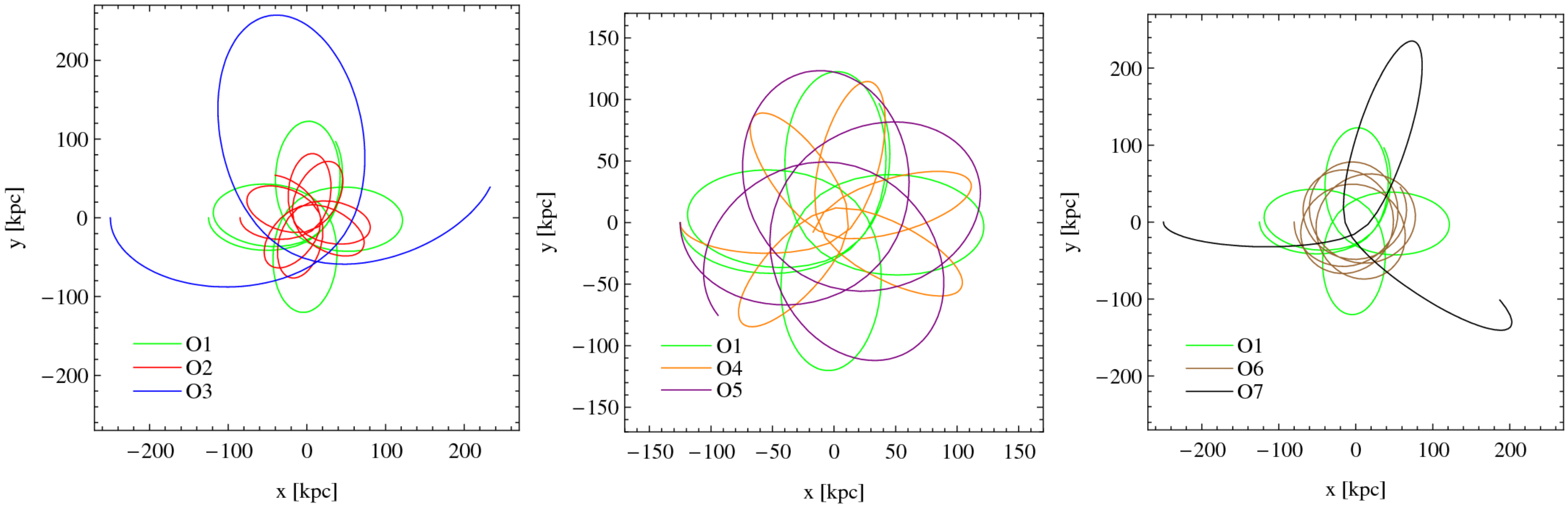
Orbit around Milky Way



- Typical orbit of a dwarf galaxy, a satellite of the Milky Way
- Time required for one radial orbit is significantly smaller than needed for one angular orbit
- The angle covered in one radial period is

$$\Delta\psi \sim 3\pi/2$$

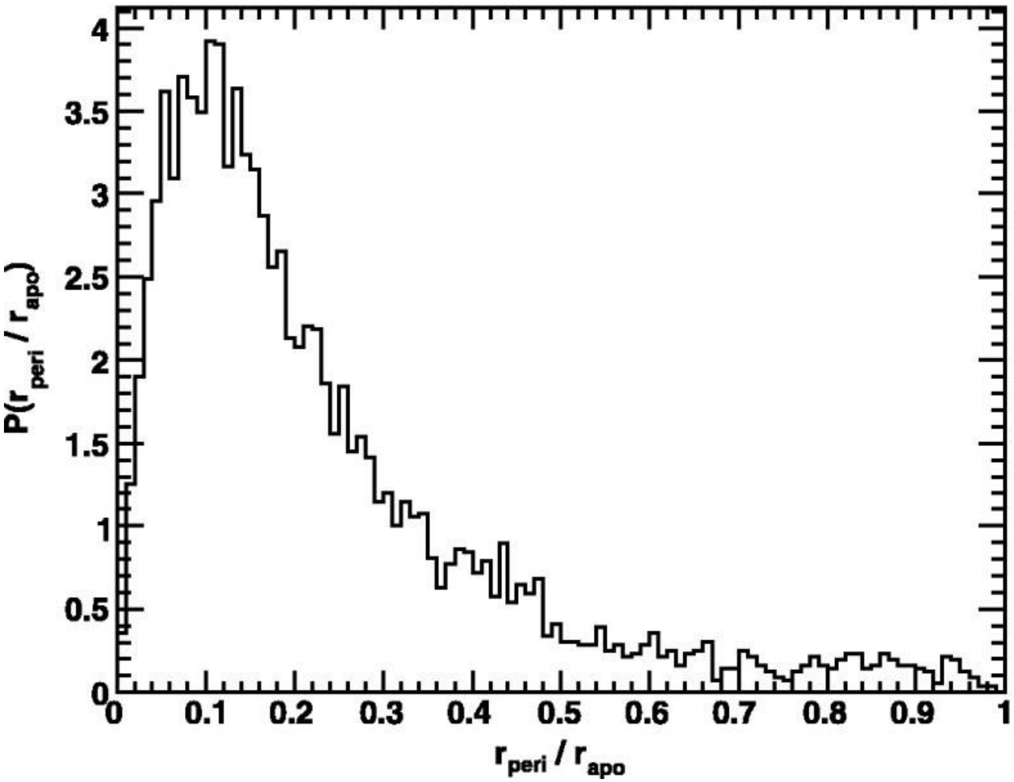
Orbits around Milky Way

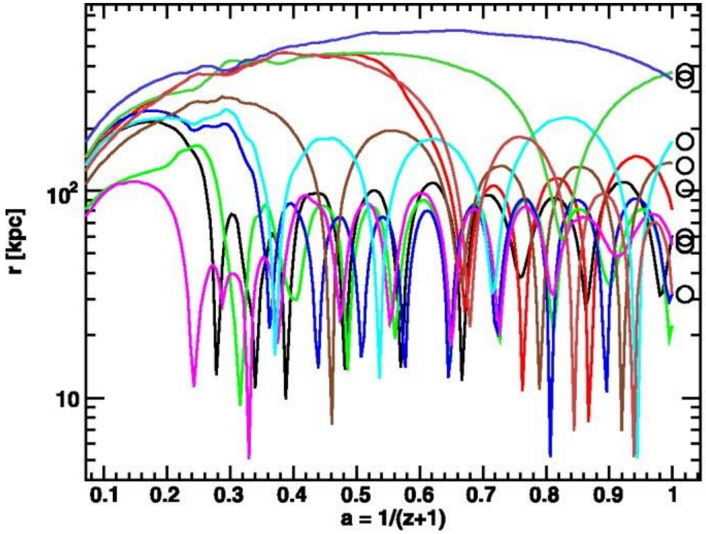


Orbit	r_{apo} [kpc]	r_{peri} [kpc]	T_{orb} [Gyr]
O1	125	25	2.09
O2	87	17	1.28
O3	250	50	5.40
O4	125	12.5	1.81
O5	125	50	2.50
O6	80	50	1.70
O7	250	12.5	4.55

- Orbital period given is usually the radial period
- For more eccentric orbits $\Delta\psi$ is lower

Distribution of orbits





Via Lactea simulation

Diemand et al. 2007

Constants and integrals of motion

- Any orbit can be described as a path in 6D space of position and velocity (phase space)
- A constant of motion is a function of phase space coordinates and time that is constant along the orbit

$$C[\mathbf{x}(t_1), \mathbf{v}(t_1), t_1] = C[\mathbf{x}(t_2), \mathbf{v}(t_2), t_2]$$

- An integral of motion is a function of phase space coordinates only that is constant along the orbit

$$I[\mathbf{x}(t_1), \mathbf{v}(t_1)] = I[\mathbf{x}(t_2), \mathbf{v}(t_2)]$$

Integrals of motion

- Orbits can have 0-5 integrals of motion
- In any static potential Φ(x) the hamiltonian is the integral of motion
- In an axisymmetric potential Φ(R,z,t) z-component of angular momentum is an integral
- In a spherical potential Φ(r,t) three components of angular momentum are integrals of motion
- More complicated integrals may exist
- The integrals reduce the dimensionality of phase space

Axisymmetric potentials

- A cylindrical coordinate system (R,z,φ) is useful in this case, with z axis aligned with the symmetry axis of the galaxy
- Orbits in the equatorial plane are the same as in the spherical potential
- Due to conservation of the z-component of the angular momentum, description of orbits can be reduced to 2D problem in R,z (meridional plane)
- The potential is of the form $\Phi(R,z)$

Equations of motion

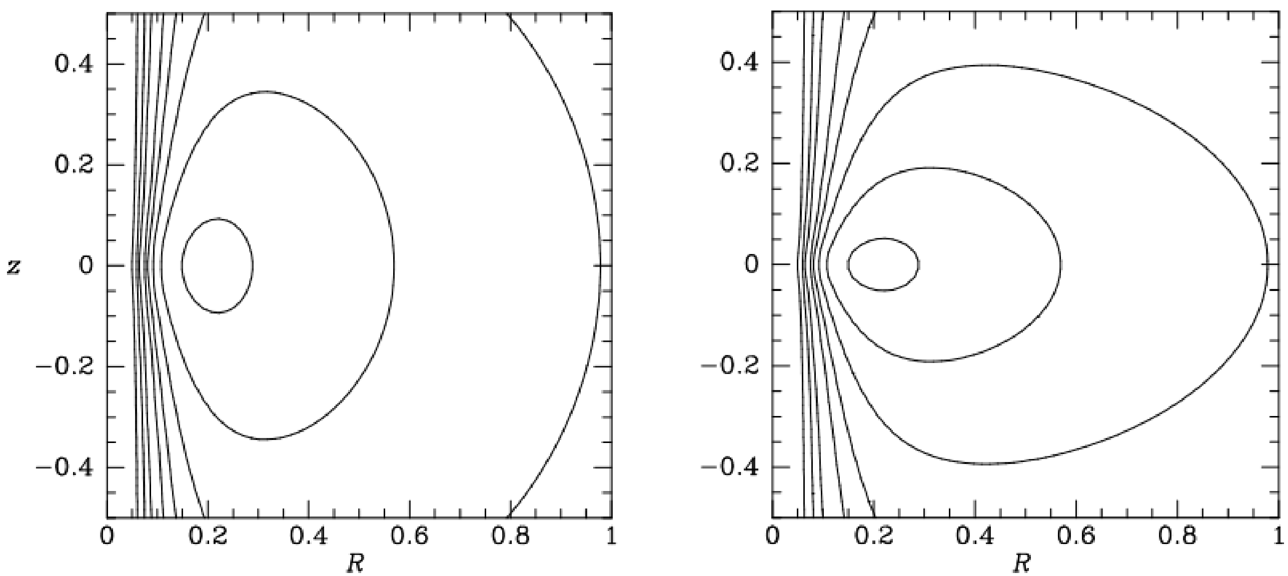
$$\ddot{R} = -\frac{\partial \Phi_{\rm eff}}{\partial R} \quad ; \quad \ddot{z} = -\frac{\partial \Phi_{\rm eff}}{\partial z},$$

$$\Phi_{\rm eff} \equiv \Phi(R, z) + \frac{L_z^2}{2R^2}$$

The effective potential defines the zero-velocity curve,

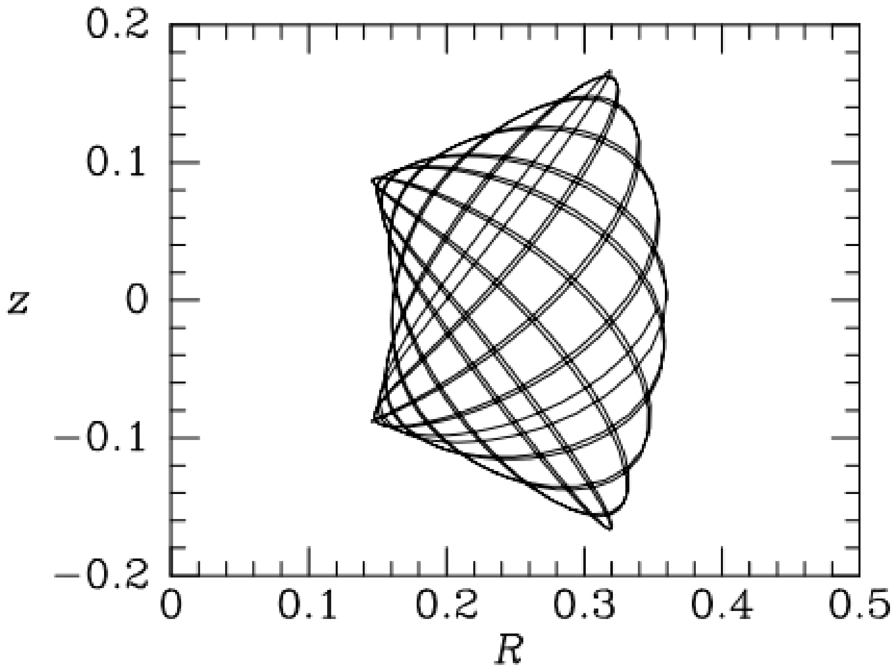
the orbit satisfies

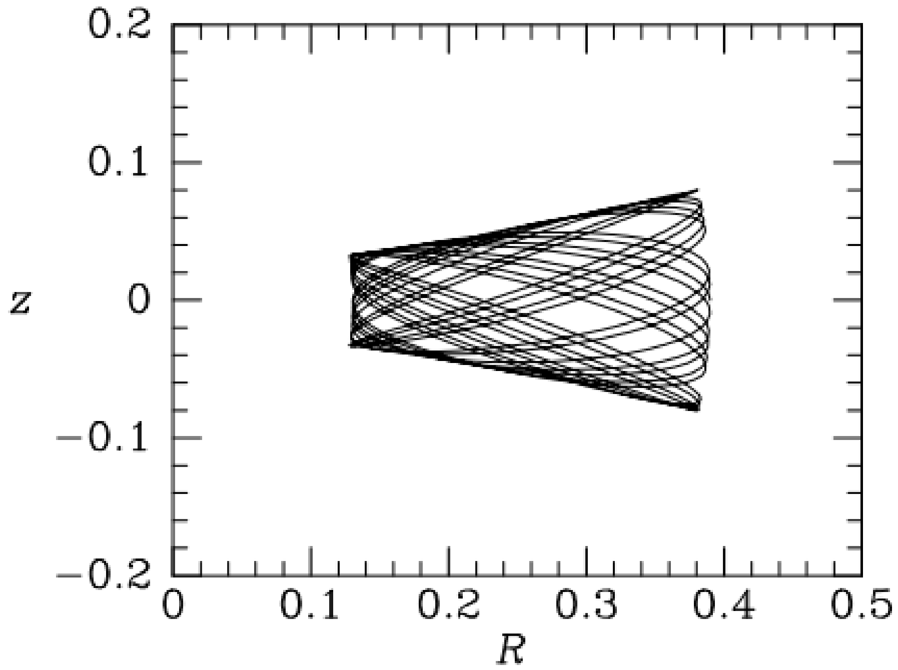
$E >= \Phi_{\rm eff}$



Examples of contours of equal effective potential

Examples of orbits





In general, the orbits must be integrated numerically, even for rather simple potentials

Nearly circular orbits

- The minimum of the effective potential corresponds to the circular orbit
- For nearly circular orbits we can expand the effective potential around the guiding center R_g which is the radius of the circular orbit
- Then $x=R-R_g$ is the new radial coordinate

$$\Phi_{\rm eff} = \Phi_{\rm eff}(R_{\rm g},0) + \tfrac{1}{2}\left(\frac{\partial^2\Phi_{\rm eff}}{\partial R^2}\right)_{(R_{\rm g},0)} x^2 + \tfrac{1}{2}\left(\frac{\partial^2\Phi_{\rm eff}}{\partial z^2}\right)_{(R_{\rm g},0)} z^2 + {\rm O}(xz^2)$$

$$\kappa^2(R_{\rm g}) \equiv \left(\frac{\partial^2\Phi_{\rm eff}}{\partial R^2}\right)_{(R_{\rm g},0)} \quad ; \quad \nu^2(R_{\rm g}) \equiv \left(\frac{\partial^2\Phi_{\rm eff}}{\partial z^2}\right)_{(R_{\rm g},0)}$$

Epicycle approximation

- The equations of motion then become

$$\ddot{x} = -\kappa^2 x,$$

$$\ddot{z} = -\nu^2 z.$$

- So x and z evolve like displacements of two harmonic oscillators with frequencies κ and ν,

  κ – epicycle or radial frequency

  ν – vertical frequency

Radial frequency

- The radial frequency can be expressed in terms of the circular frequency

$$\Omega^2(R) = \frac{1}{R}\left(\frac{\partial\Phi}{\partial R}\right)_{(R,0)} = \frac{L_z^2}{R^4}, \quad \kappa^2(R_\mathrm{g}) = \left(R\frac{\mathrm{d}\Omega^2}{\mathrm{d}R} + 4\Omega^2\right)_{R_\mathrm{g}}$$

- Near the center of the galaxy, where circular speed $v_c \sim R$, $\Omega \sim v_c/R \sim$ const and $\kappa = 2\,\Omega$
- In the outer parts Ω declines with radius, at most like in the Kepler case $\Omega \sim R^{-3/2}$ and then $\kappa = \Omega$, so in general $\Omega < \kappa < 2\,\Omega$

Oort constants

$$A(R) \equiv \tfrac{1}{2}\left(\frac{v_c}{R} - \frac{\mathrm{d}v_c}{\mathrm{d}R}\right) = -\tfrac{1}{2}R\frac{\mathrm{d}\Omega}{\mathrm{d}R},$$

$$B(R) \equiv -\tfrac{1}{2}\left(\frac{v_c}{R} + \frac{\mathrm{d}v_c}{\mathrm{d}R}\right) = -\left(\Omega + \tfrac{1}{2}R\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)$$

$$\Omega = A - B \quad ; \quad \kappa^2 = -4B(A - B) = -4B\Omega.$$

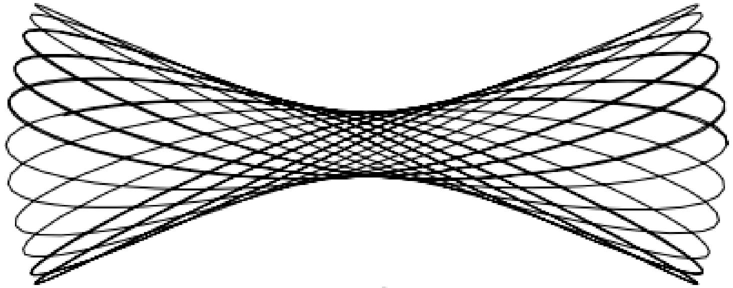
The values of the constants can be measured directly from the kinematics of stars in the solar neighbourhood, the measurements give:

$$\kappa_0 / \Omega_0 = 1.35$$

so the Sun makes 1.35 oscillations in the radial direction when it goes around galactic center

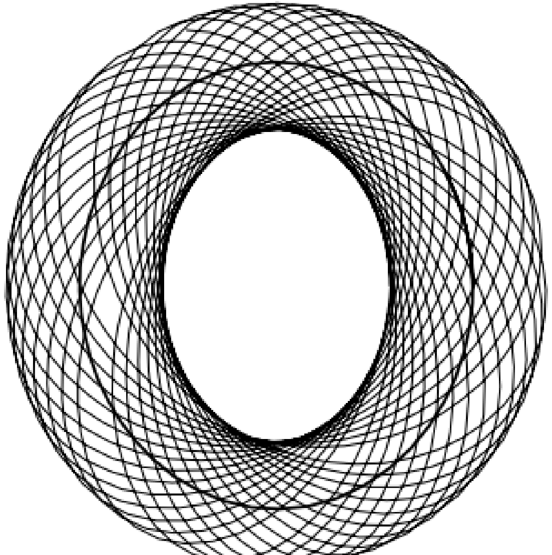
Flat, non-axisymmetric potentials

- For example: a bar in a thin disk
- Two types of orbits are essentially possible:



box orbits

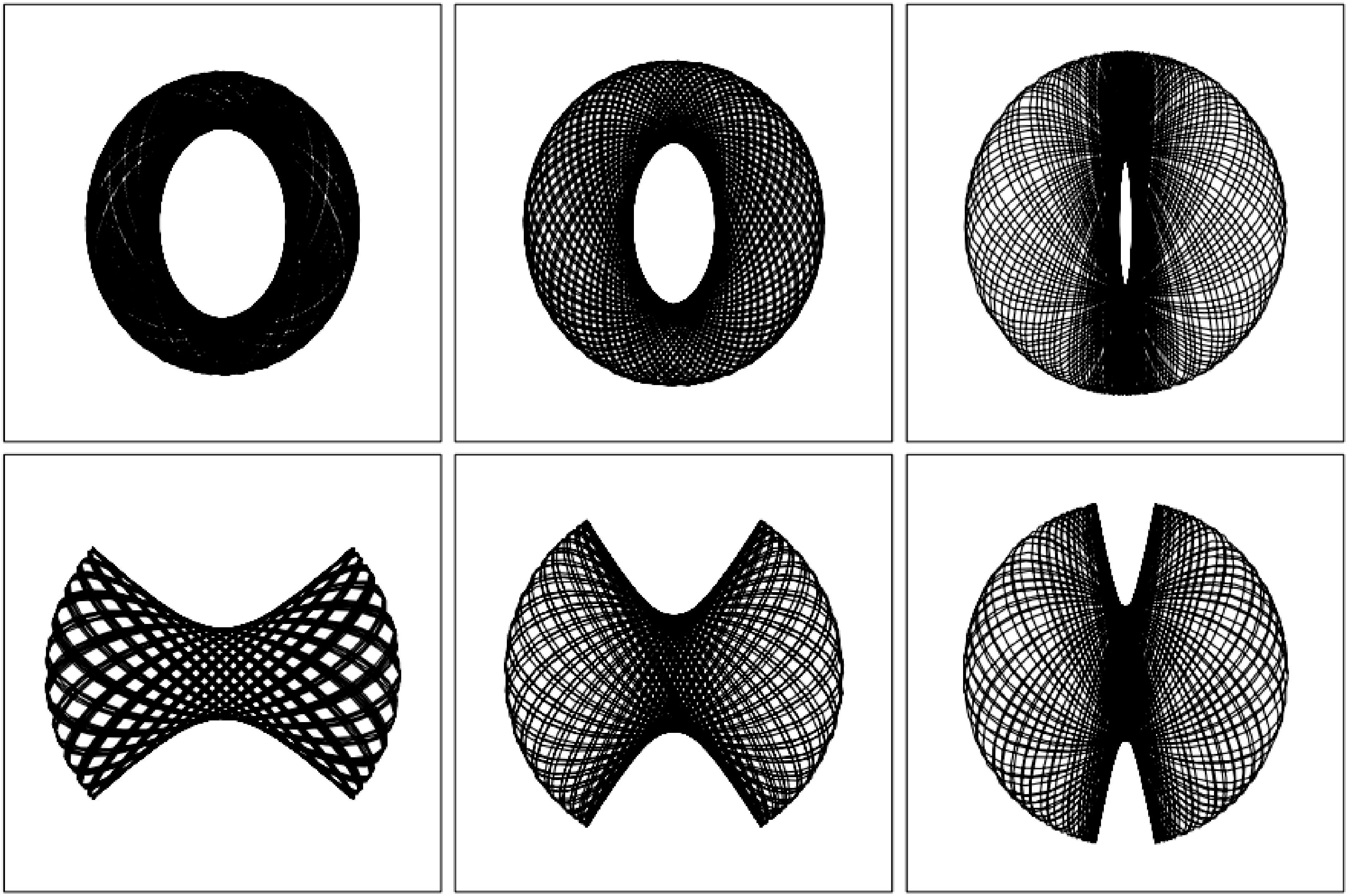
(combination of two harmonic oscillators, go through the center)



loop orbits

(circulating in a fixed direction but oscillating in radius, never go through the center)

Different shapes

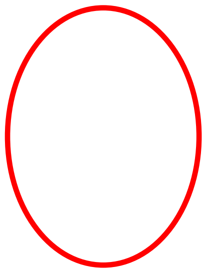


loop

box

Parenting

- Special among the loop and box orbits are the closed orbits which are stable
- A closed loop orbit is the parent of the family of all loop orbits



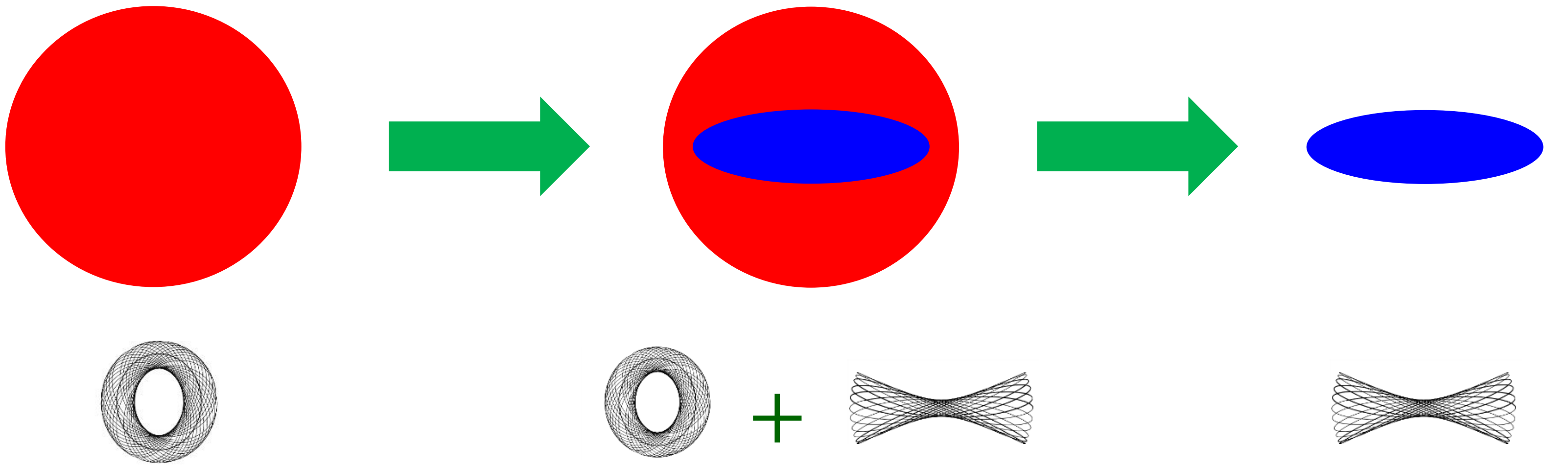
- A closed long-axis orbit is the parent of box orbits

- Any member of a given family of orbits may be viewed as performing stable oscillations about the parent closed orbit

The limit of axisymmetry

- In the 2D axisymmetric potential there are only two stable closed orbits for each energy: clockwise and anti-clockwise circular orbits
- All other orbits belong to families parented by these two orbits
- The epicycle frequency is the frequency of small oscillations around the parent closed orbit

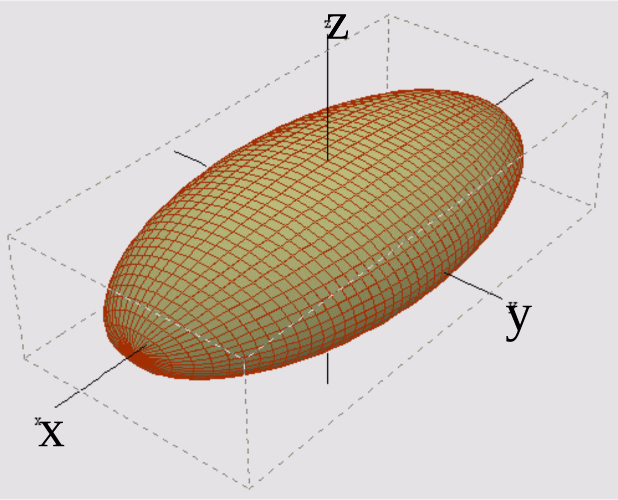
From disk to bar



- The idea of stable closed orbits allows us to study the evolution of orbital structure when potential changes
- For disks only loop orbits are present
- When the bar appears, long-axis orbit becomes stable and parents the box orbits

Triaxial potentials

- Most elliptical galaxies can be described as triaxial
- Additional complication is introduced by the presence of a massive black hole
- The simplest model is the so-called perfect ellipsoid (de Zeeuw 1985)

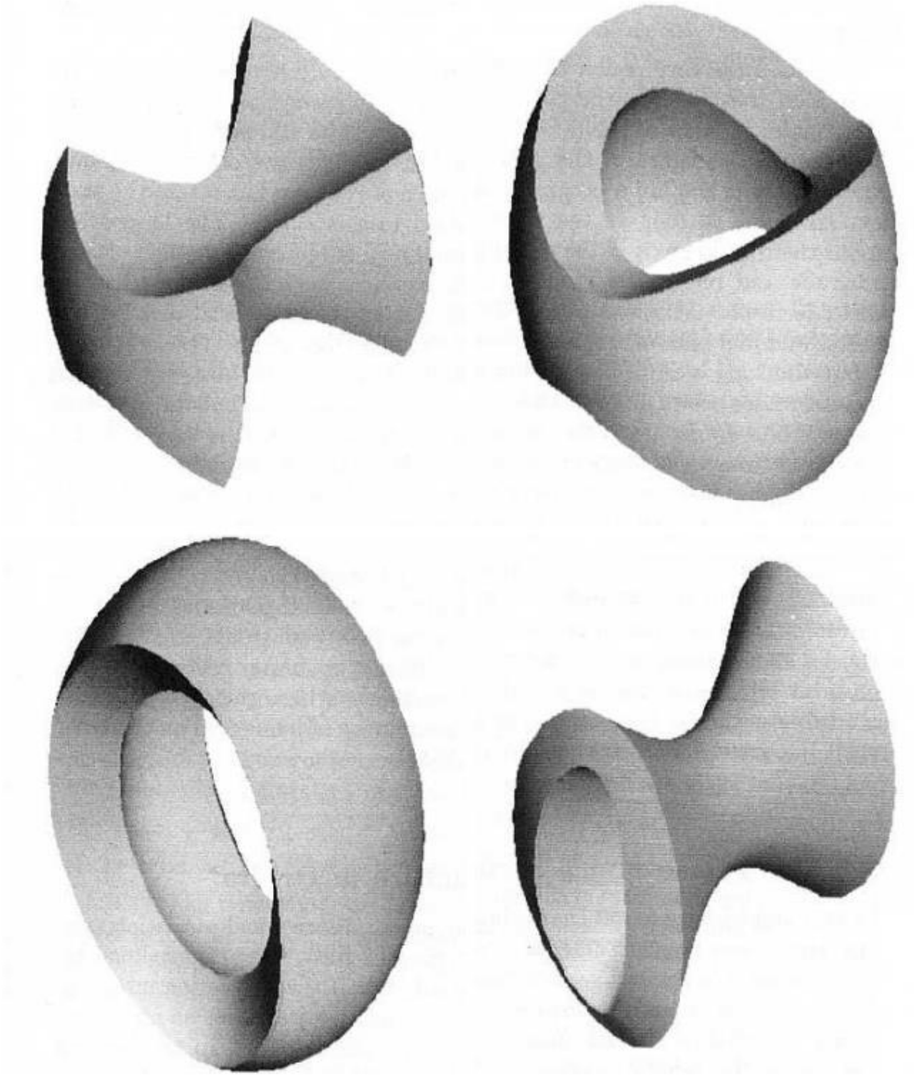


$$\rho(\mathbf{x}) = \frac{\rho_0}{(1+m^2)^2} \quad \text{where} \quad m^2 \equiv \frac{x^2 + (y/q_1)^2 + (z/q_2)^2}{a_0^2}$$

Four families of orbits

box

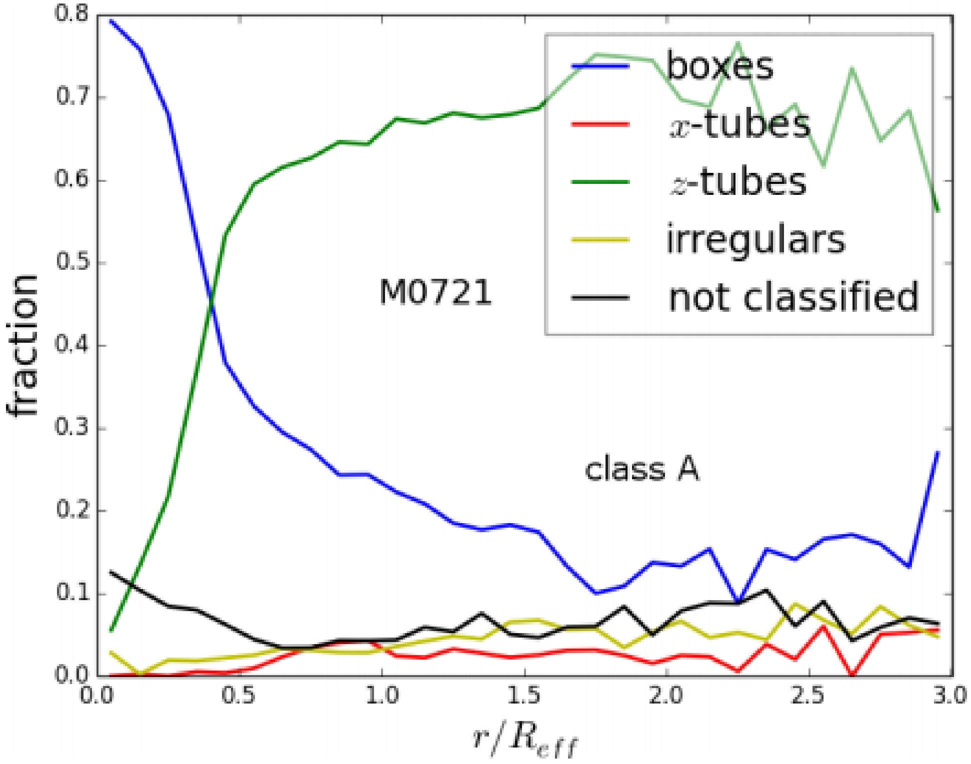
short-axis tube
(z-tube)



outer long-
axis tube
(x-tube)

inner long
axis tube
(x-tube)

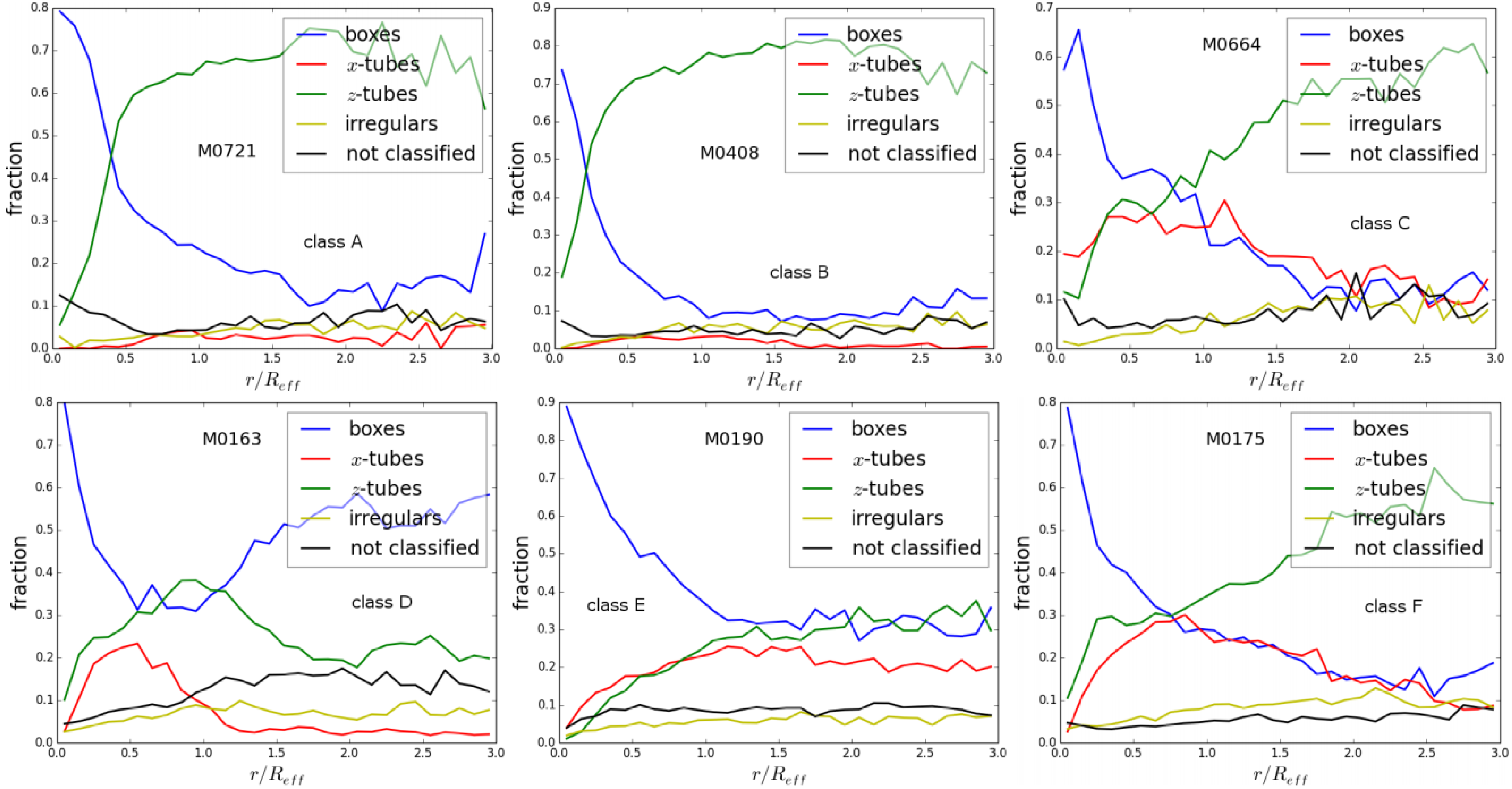
Orbits in simulated galaxies



Rottgers et al. 2014

- In simulated galaxies we can use the full 3D information of the motions of the stars to classify the orbits
- The variety of orbits is richer than the four classes of regular (periodic) orbits
- Fraction of different orbits as a function of radius can be measured

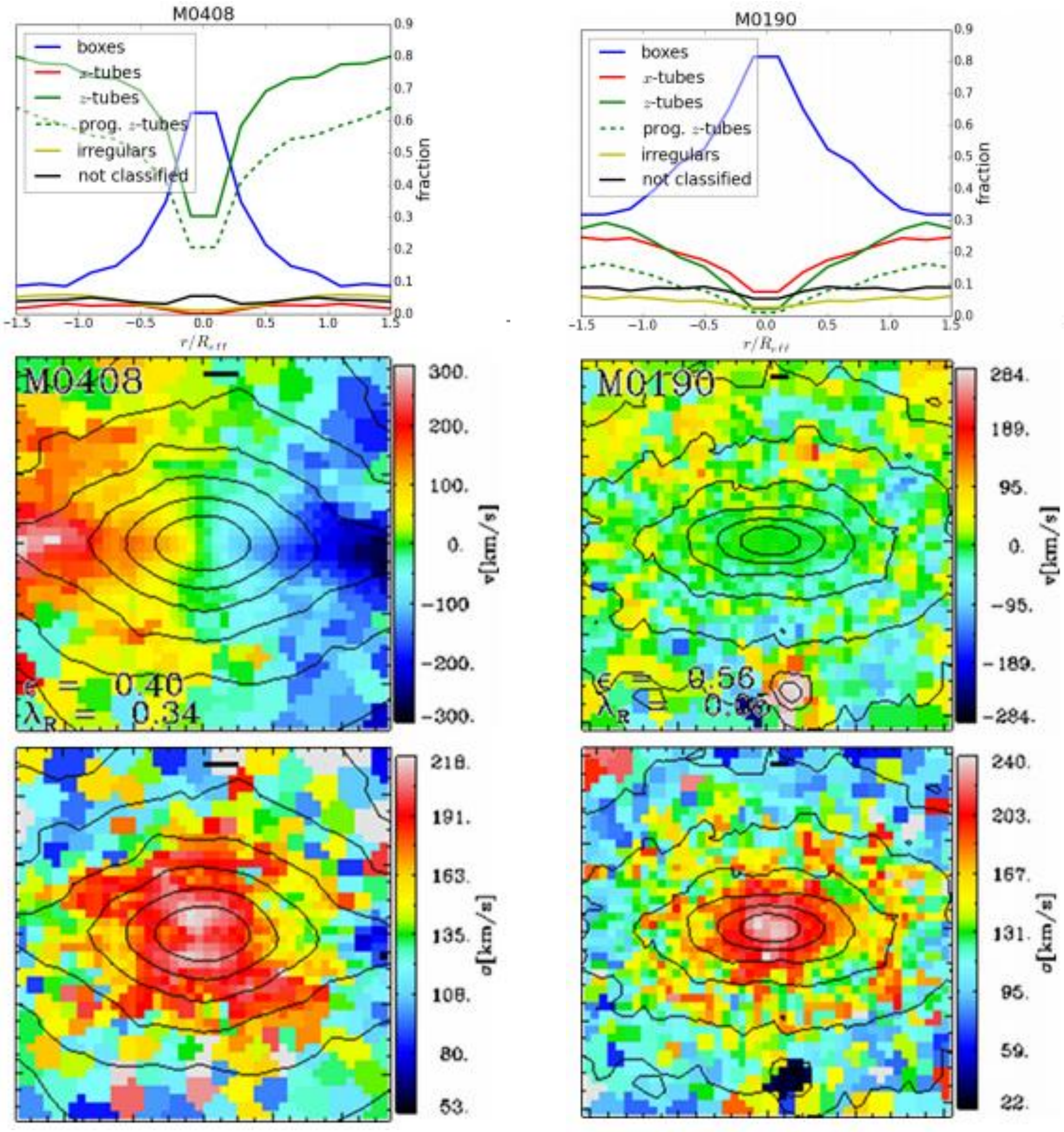
Orbits in simulated galaxies



Rottgers et al. 2014

Galaxies of classes A-F have different formation history

Line-of-sight kinematics



z-tubes dominate   boxes dominate

- Only projections along one direction are accessible to observation
- We must deduce the orbital composition from these limited data

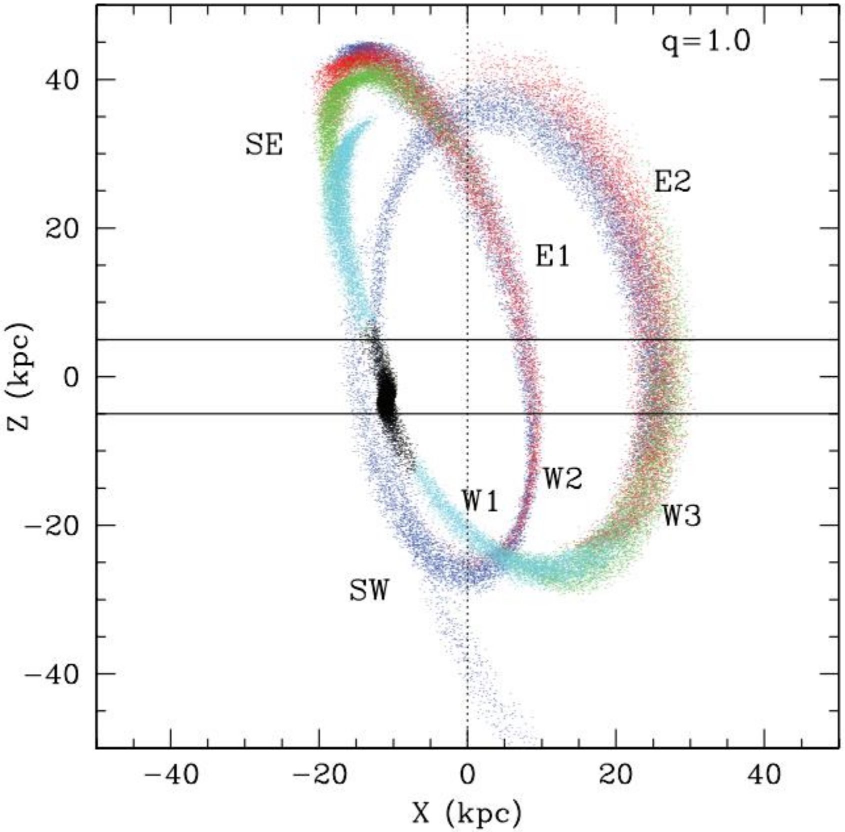
Rottgers et al. 2014

Orbits as probes of potential

NGC 5907

Stellar streams originating from the tidally stripped dwarf galaxies or globular clusters can be used to study the potential of their hosts

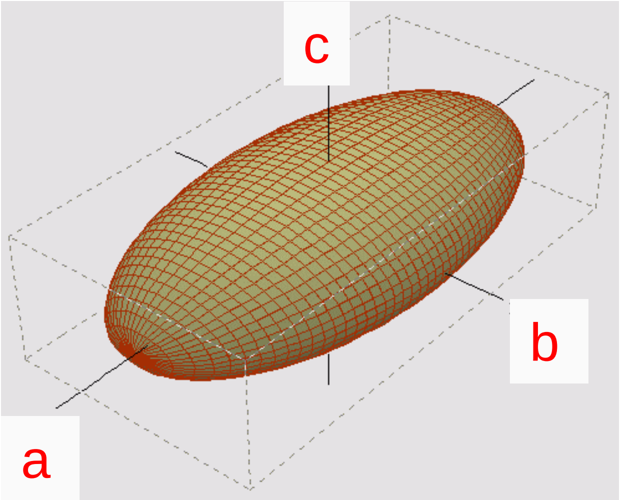
NGC 5907



Martinez-Delgado et al. 2008

Shapes of dark matter haloes

- Since the dark matter haloes are the most massive components of galaxies they influence the orbits most
- The expected shapes of dark matter haloes from cosmological simulations are significantly different from spherical
- The shapes of the haloes can be approximated as triaxial ellipsoids

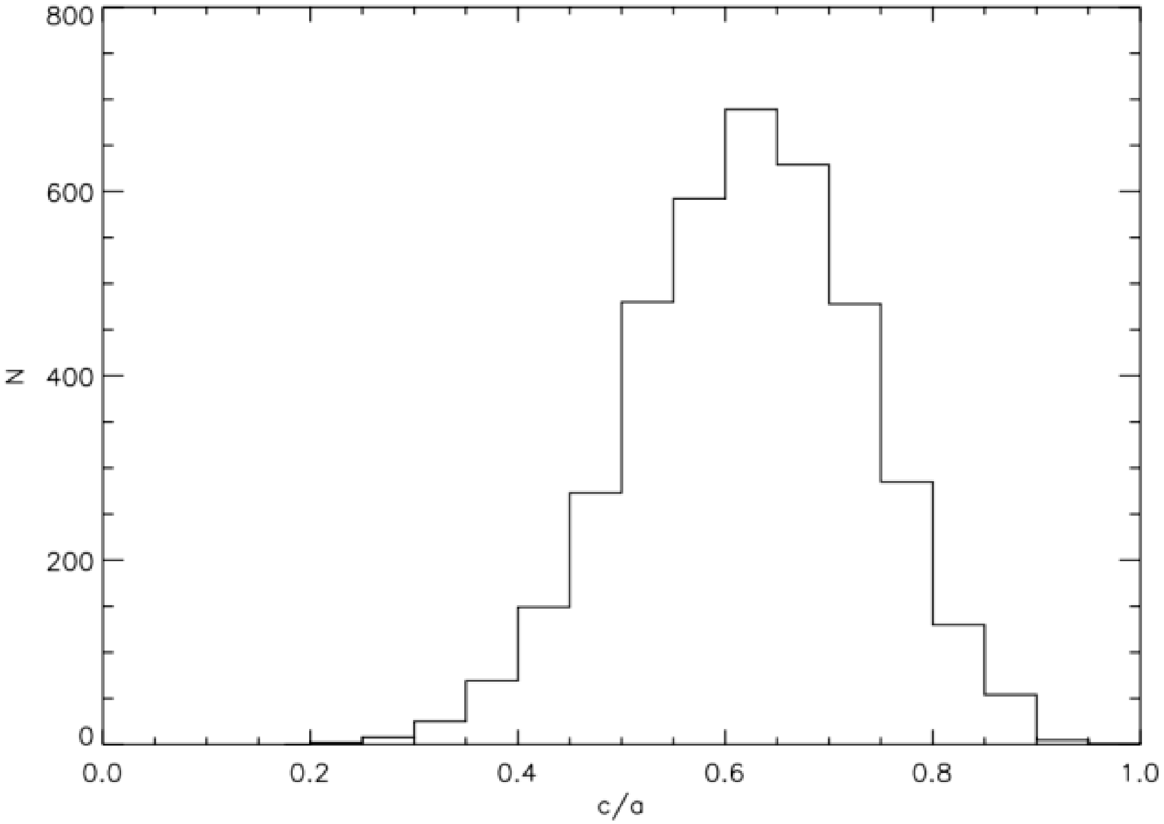


a – longest axis

b – intermediate axis

c – shortest axis

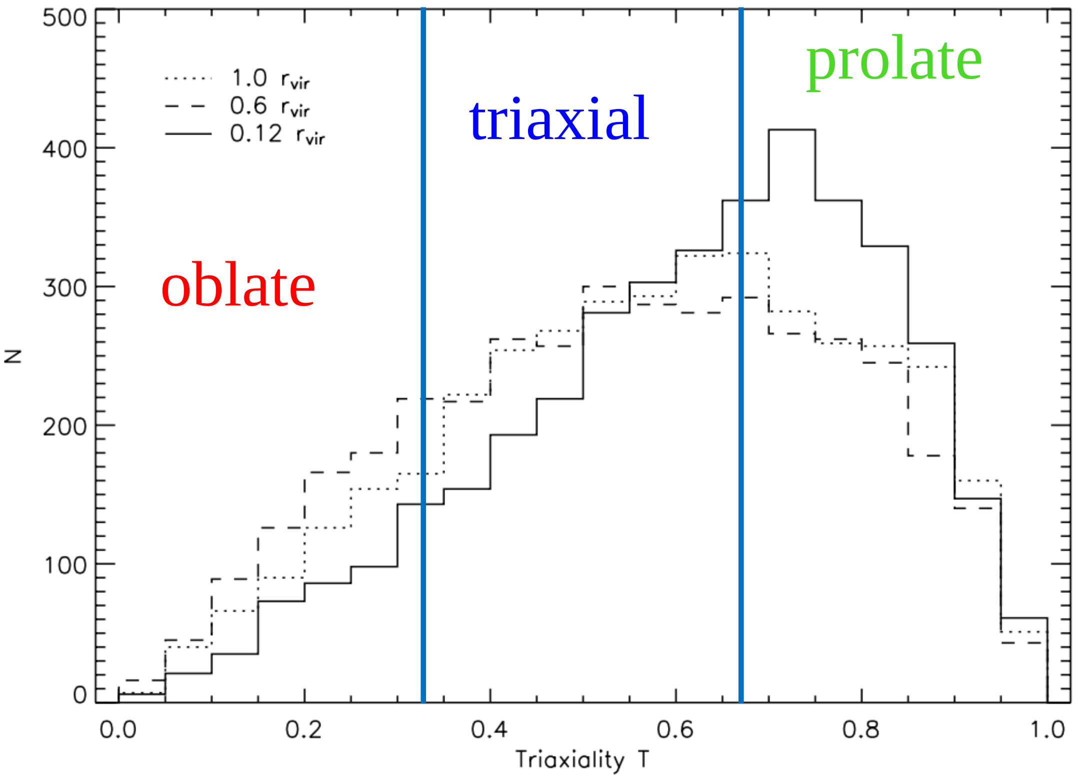
Shapes of dark haloes



The distribution of axis ratio c/a measured at 0.4 r_{vir} for ~4000 haloes of mass 10^{11} -10^{14} $M_{\odot}$ identified in a simulation of structure formation in a box of size ~50 Mpc

Bailin & Steinmetz 2005

Triaxiality



A good measure of shape is the triaxiality parameter

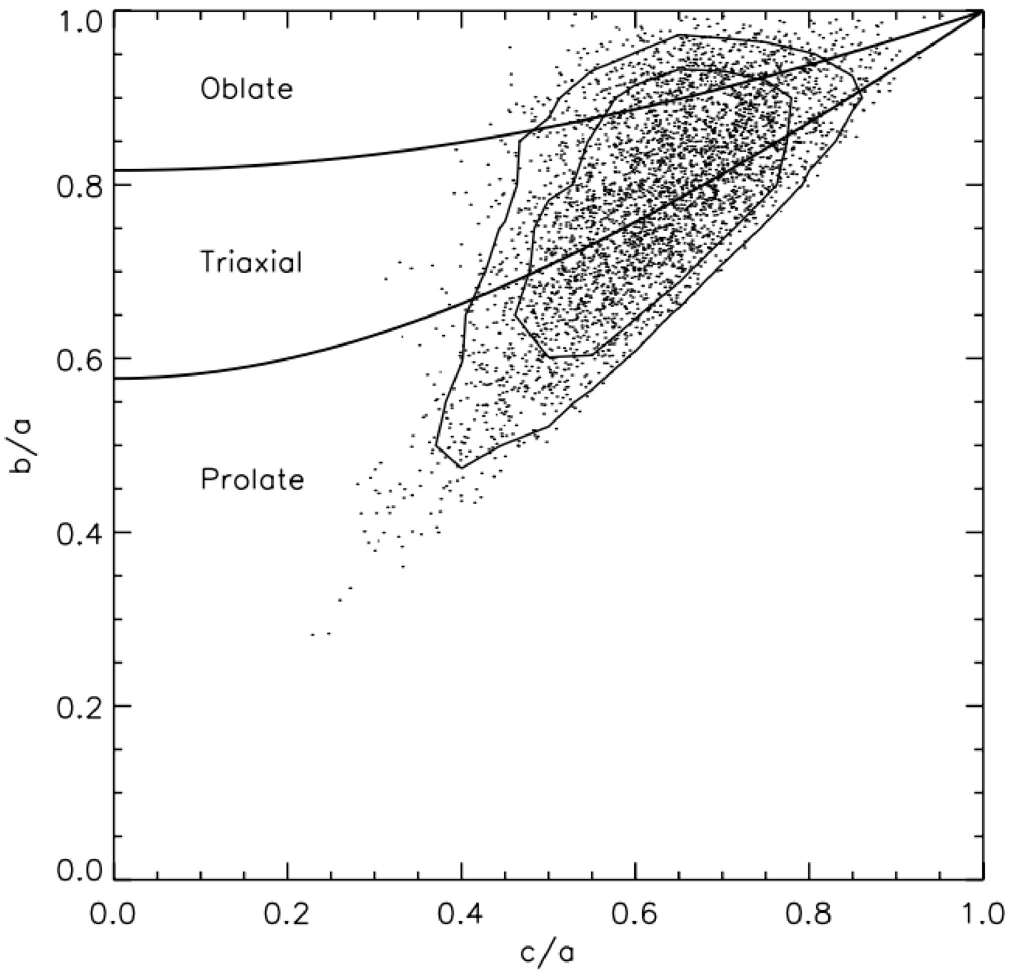
$$T = \frac{a^2 - b^2}{a^2 - c^2}$$

0 < T < 1/3	oblate
1/3 < T < 2/3	triaxial
2/3 < T < 1	prolate

The shape depends on radius

Bailin & Steinmetz 2005

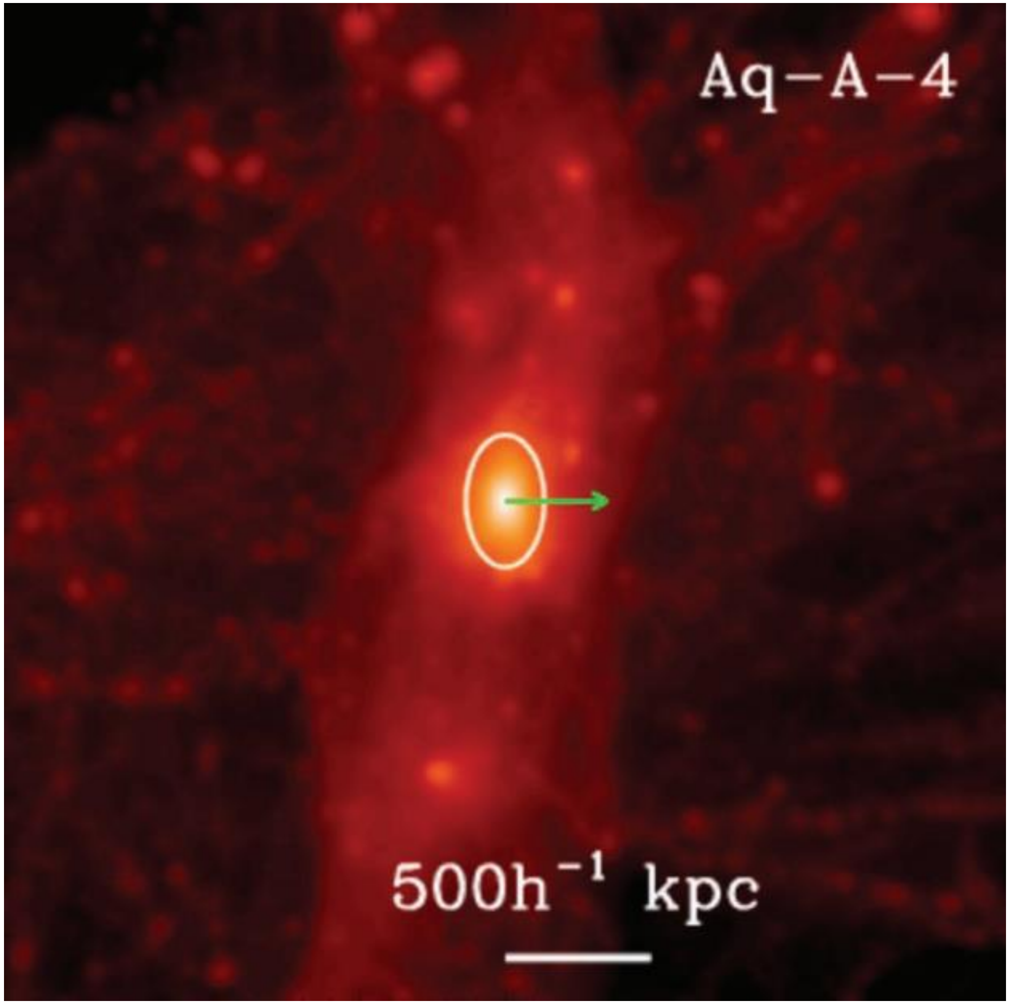
Axis ratios



Bailin & Steinmetz 2005

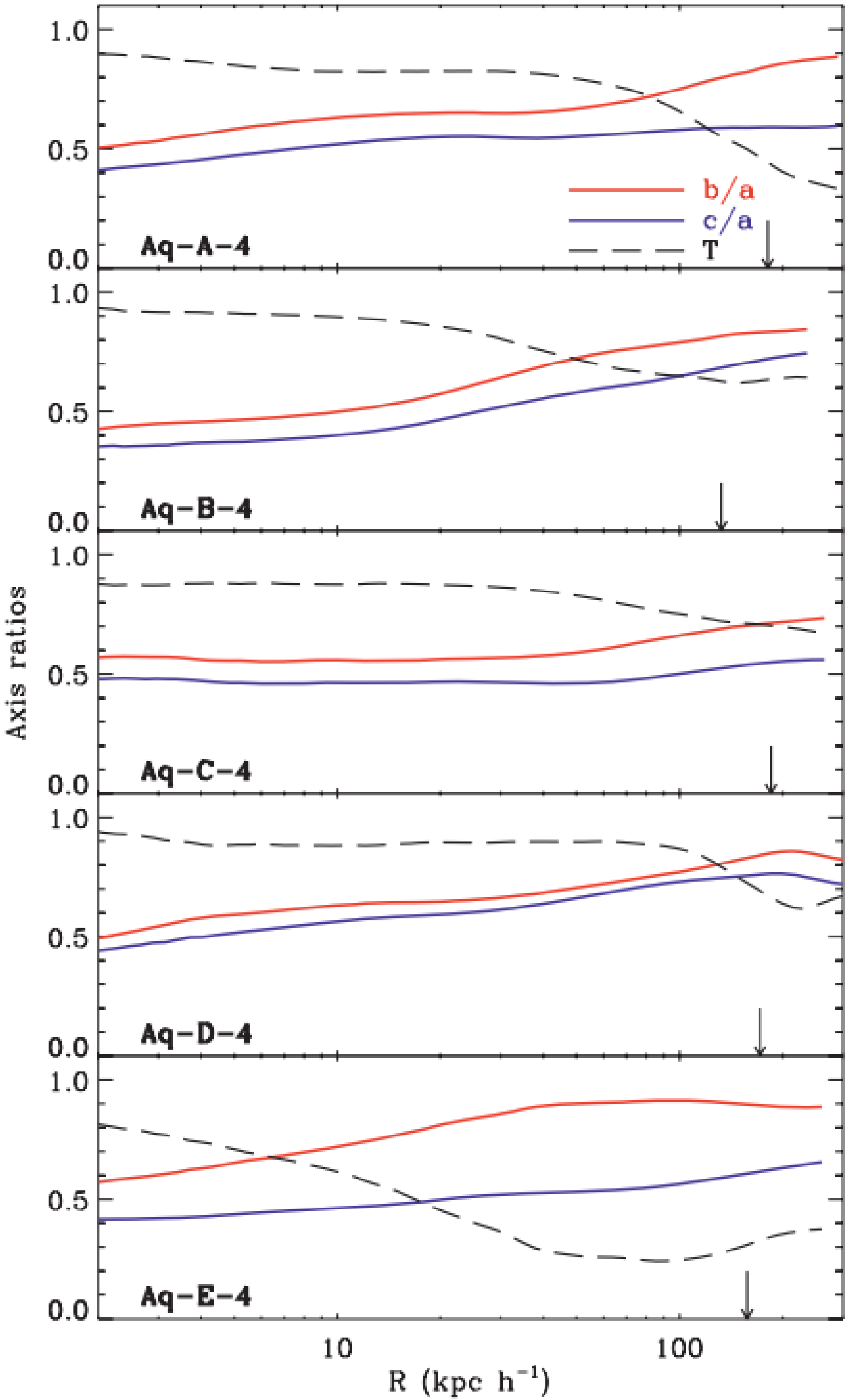
- The distribution of halo properties in the c/a-b/a plane
- Most haloes are triaxial with c/a~0.6 and b/a~0.8
- There are more prolate than oblate haloes
- There is some dependence on halo mass (massive haloes are more flattened)

Dark halo of the Milky Way



Vera-Ciro et al. 2011

- An example of a Milky Way-like halo formed in simulations
- The halo is strongly non-spherical
- The shape depends on the radius where we measure it

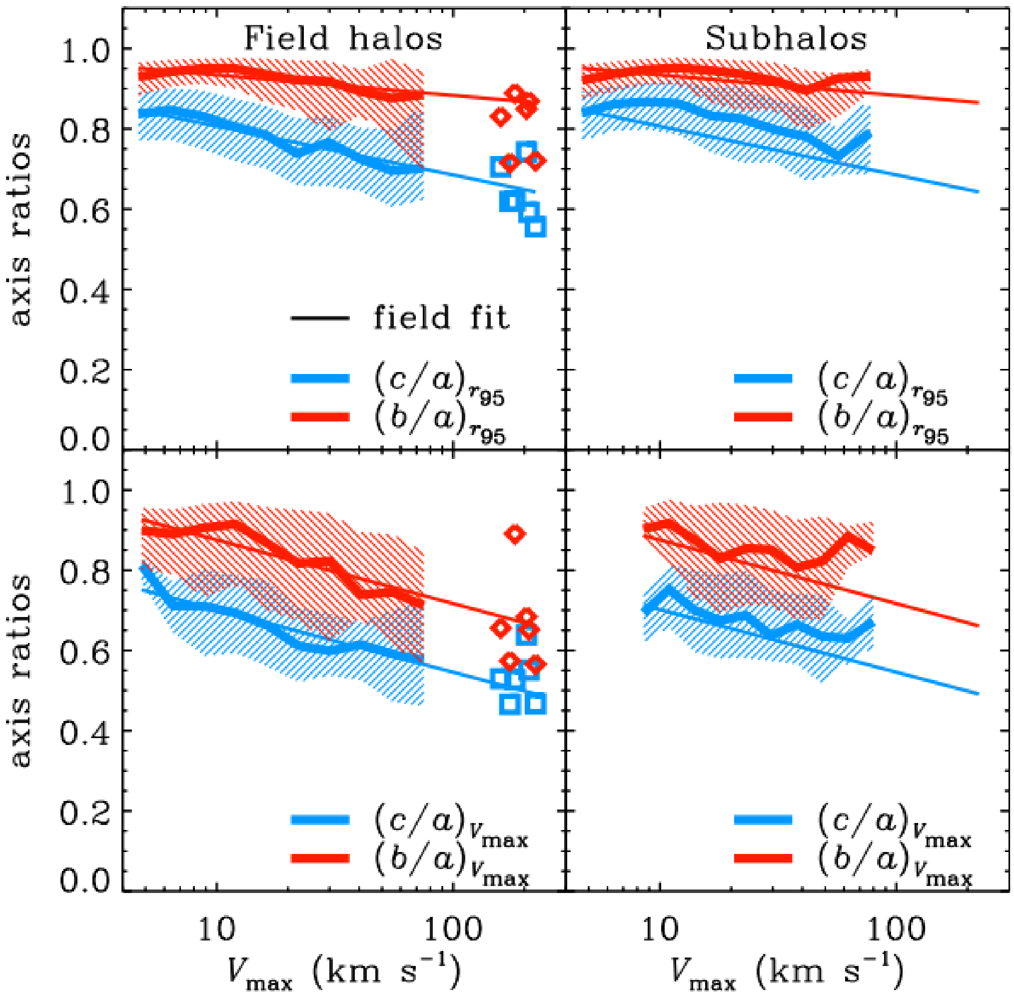


Dependence of shape on radius

- The axis ratios grow with radius
- The Milky Way halo is more spherical in the outer parts
- The same is true for different realizations of Milky Way halo

Vera-Ciro et al. 2011

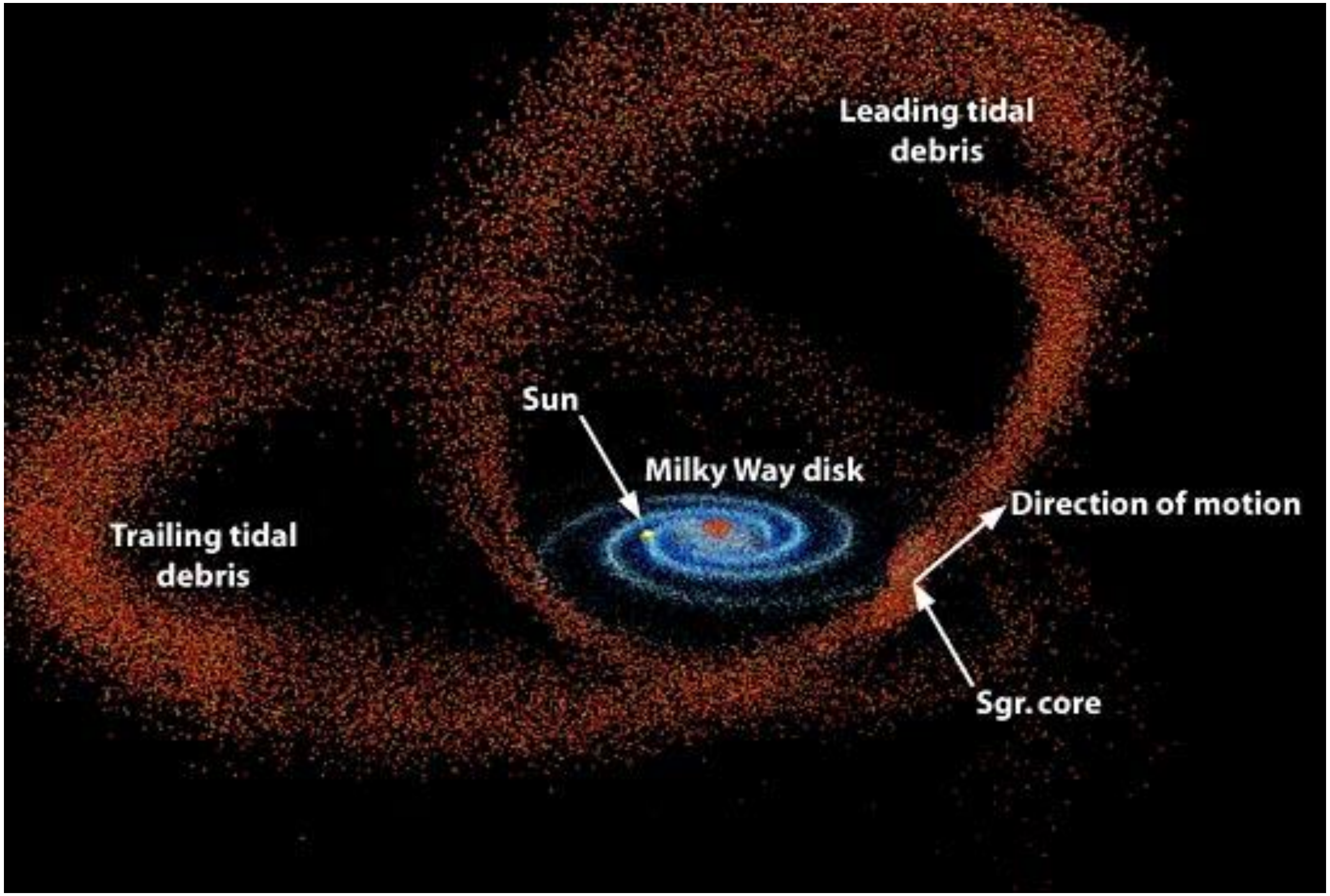
Dependence of shape on mass



- The axis ratios decrease with mass
- Less massive haloes are more spherical
- The dependence on mass is extended to lower masses
- The effect is similar in all environments

Vera-Ciro et al. 2014

Sagittarius stream



The best studied example is the modelling of the Sagittarius stream

A model for the Milky Way

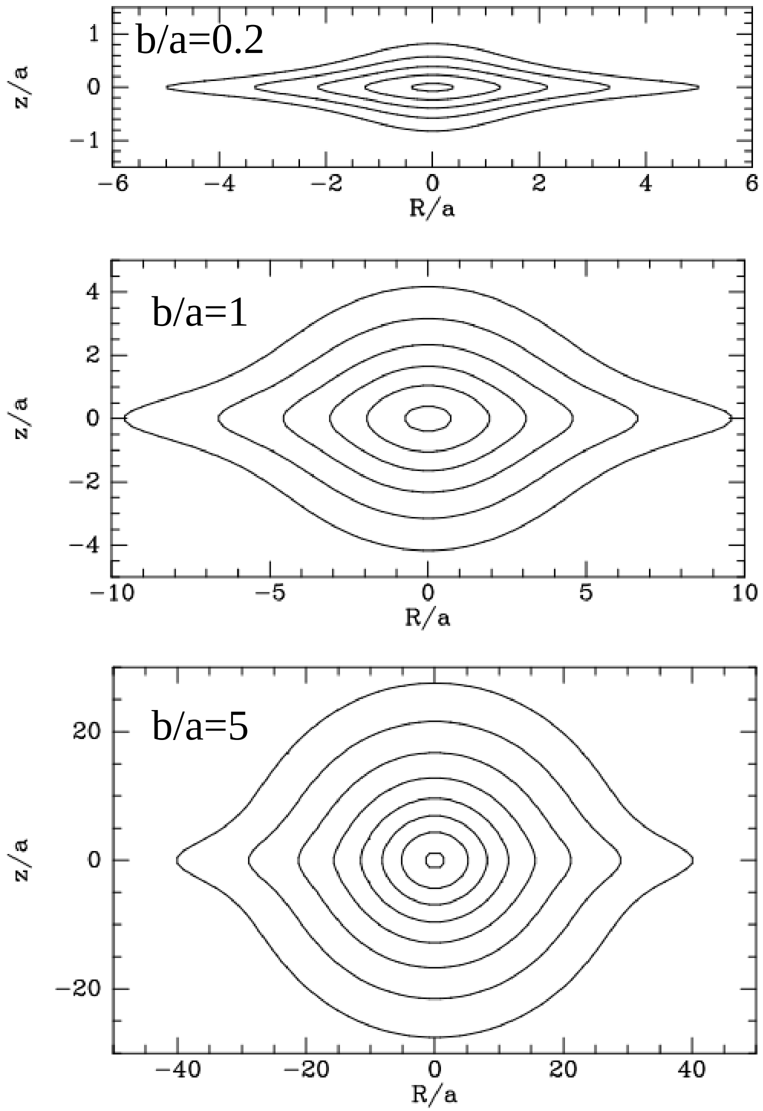
$$\Phi_{\text{disk}} = -\alpha \frac{G M_{\text{disk}}}{\sqrt{R^2 + (a + \sqrt{z^2 + b^2})^2}},$$

$$\Phi_{\text{sphere}} = -\frac{G M_{\text{sphere}}}{r + c},$$

$$\Phi_{\text{halo}} = v_{\text{halo}}^2 \ln \left(C_1 x^2 + C_2 y^2 + C_3 x y + (z/q_z)^2 + r_{\text{halo}}^2 \right)$$

- The MW model is composed of a Miyamoto-Nagai disk, a Hernquist bulge and a logarithmic dark matter halo
- The halo is described by an ellipsoid that can be rotated about the MW z-axis

Miyamoto-Nagai models



$$\Phi_M(R,z) = -\frac{GM}{\sqrt{R^2 + (a + \sqrt{z^2+b^2})^2}}$$

$$\rho_M(R,z) = \left(\frac{b^2 M}{4\pi}\right) \frac{aR^2 + (a + 3\sqrt{z^2+b^2})(a+\sqrt{z^2+b^2})^2}{\left[R^2 + (a+\sqrt{z^2+b^2})^2\right]^{5/2} (z^2+b^2)^{3/2}}$$

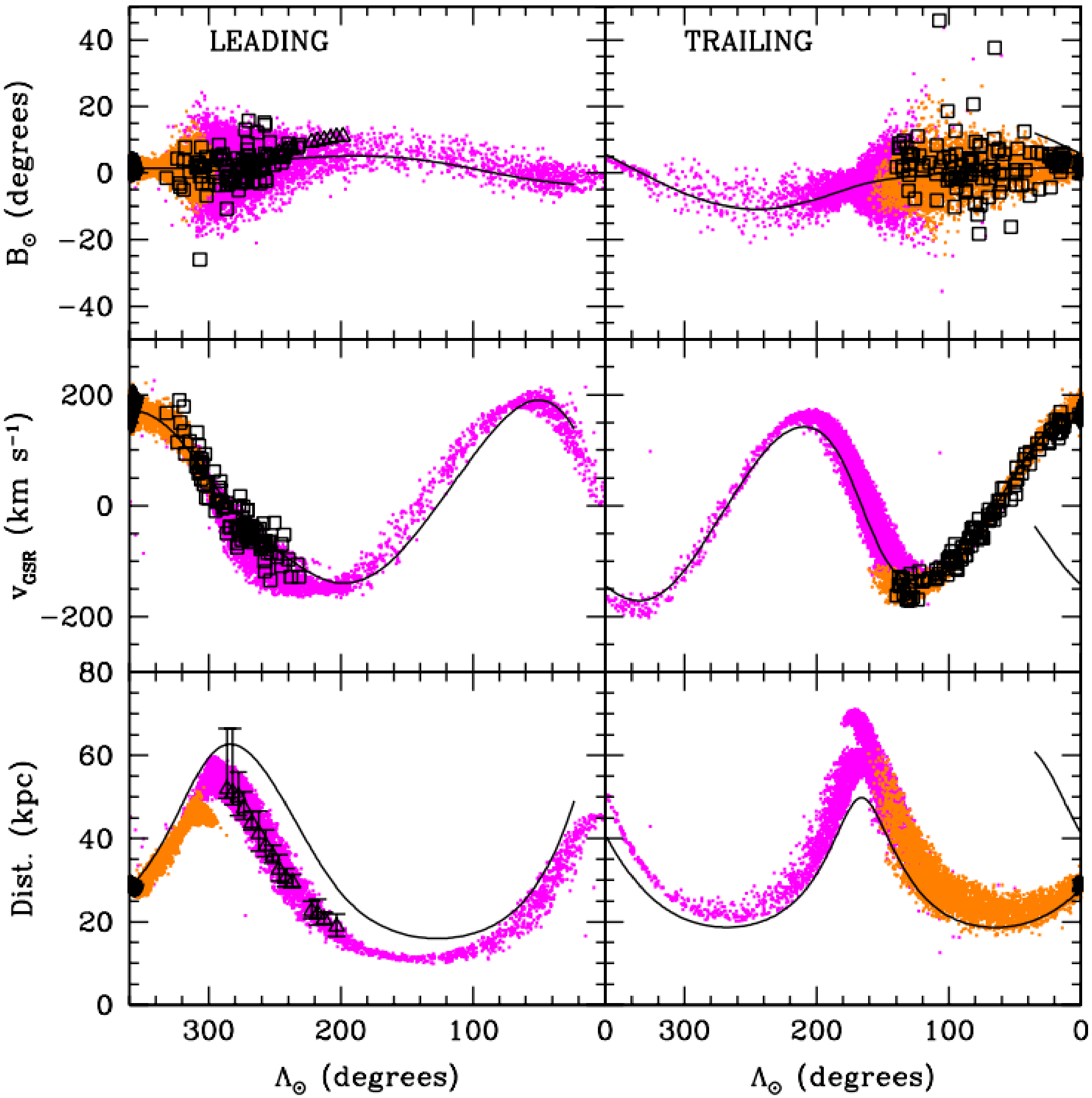
Potential-density pair in analytic form

b → 0 thin disk

a → 0 Plummer model

Miyamoto & Nagai 1975

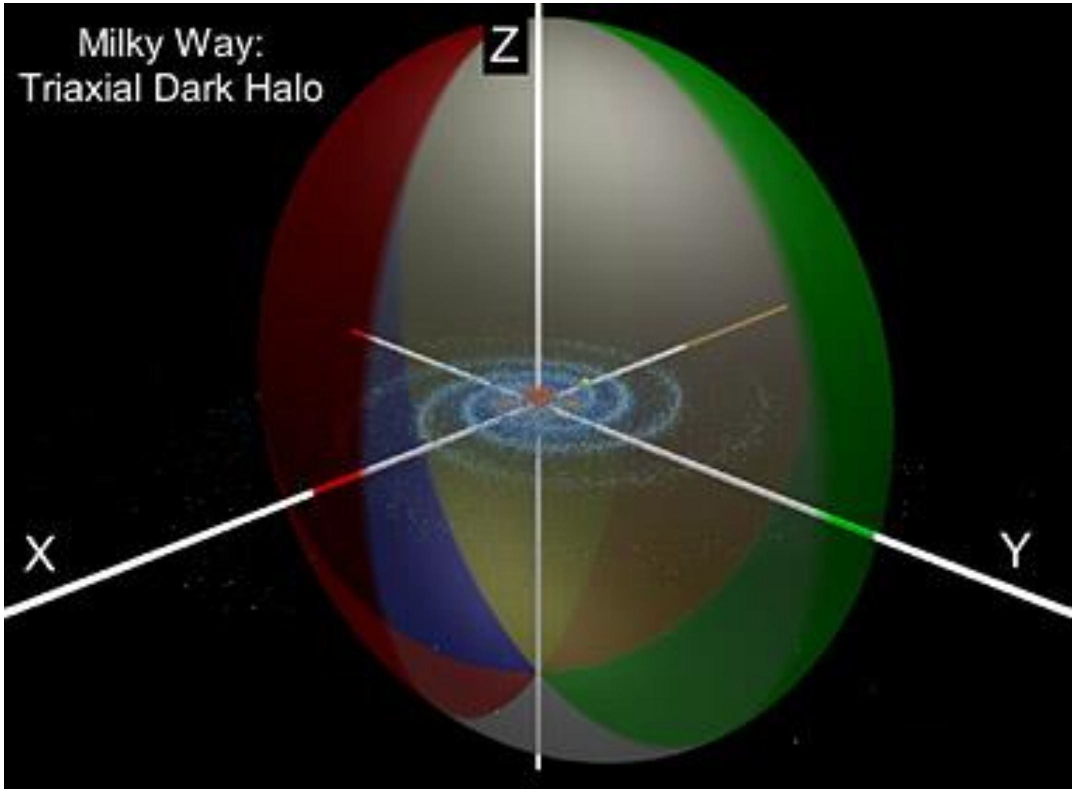
Model for Sgr stream



- N-body model vs data (black points)
- Debris from the last 3 Gyr shown
- Black line shows the trajectory of Sgr core
- Best fit found for non-spherical halo

Law & Majewski 2010

Oblate dark matter halo

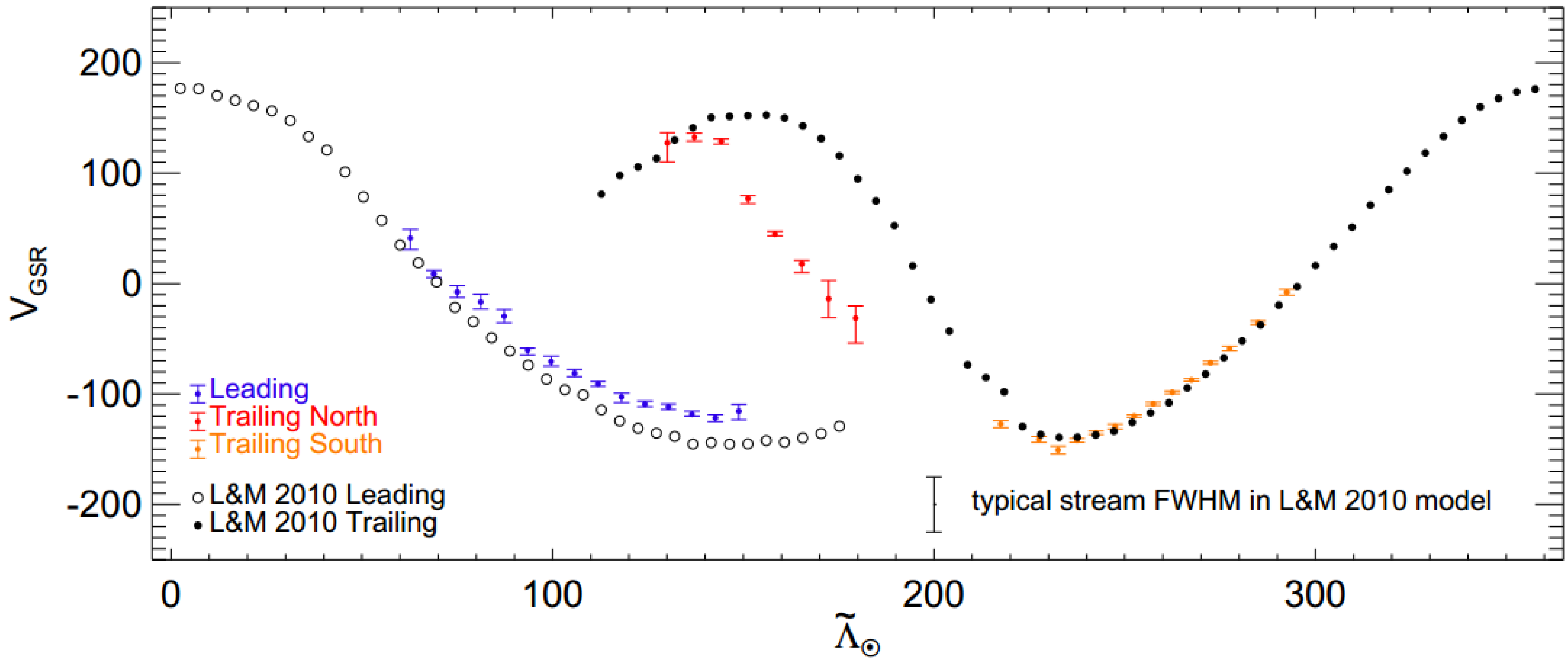


- The best-fitting dark halo of the Milky Way is flattened
- The shape can be approximated by an almost oblate ellipsoid with c/a=0.72 and b/a=0.99

- The minor axis of the halo lies in the plane of the Milky Way disk

Law & Majewski 2010

Newer data



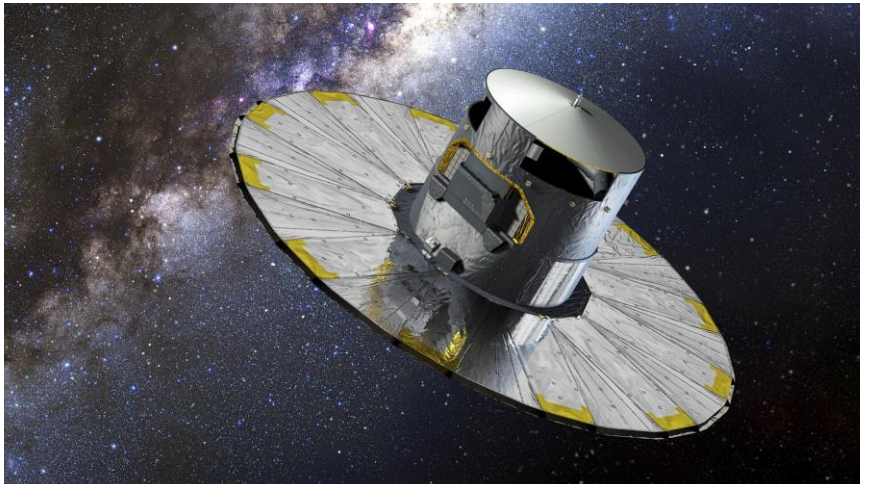
Belokurov et al. 2014

The model of Law & Majewski does not match the new data

Orbits of dwarfs in the Local Group

- Determination of orbits of dwarfs in the Local Group is difficult because we need to measure the proper motion
- Proper motion measurements have been done for a many dwarfs including classical and ultra-faint objects
- This was possible

  due to Gaia satellite



www.esa.int/Science_Exploration/Space_Science/Gaia

Orbits of dwarfs

Galaxy	Peri(light) [kpc]	Apo(light) [kpc]
Antlia II	54^{+10}_{-8}	158^{+11}_{-10}
Aquarius II	75^{+26}_{-50}	124^{+180}_{-15}
Bootes I	42^{+11}_{-10}	108^{+36}_{-20}
Bootes II	$38.7^{+2.0}_{-1.6}$	–
Bootes III	$8.6^{+2.0}_{-2.0}$	188^{+47}_{-23}
Canes Venatici I	68^{+72}_{-42}	301^{-}_{-28}
Canes Venatici II	49^{+61}_{-33}	243^{-}_{-42}
Carina	$106.7^{+6.4}_{-5.4}$	248^{-}_{-110}
Carina II	$28.2^{+2.7}_{-2.2}$	–
Carina III	$28.7^{+1.2}_{-1.3}$	–
Columba I	$183.3^{+7.7}_{-10.1}$	–
Coma Berenices	$46.0^{+1.9}_{-3.6}$	165^{+100}_{-48}
Crater I	100^{+45}_{-73}	152^{-}_{-11}
Crater II	$39.1^{+8.0}_{-7.7}$	149^{+11}_{-10}
Draco	$51.7^{+4.1}_{-6.1}$	138^{+14}_{-17}
Draco II	$15.0^{+5.1}_{-0.6}$	191^{+74}_{-39}

Orbital parameters were obtained by integration of orbits in an assumed potential of the Milky Way

Battaglia et al. 2022

Collisionless systems

- We will assume that galaxies are composed of N identical point masses
- Stellar systems can be approximated as collisionless: the stars move as if the system’s mass was smoothly distributed in space rather than in form of point-like masses
- Rather than following the orbits of all stars we will describe the system in terms of the distribution function

The distribution function

- The distribution function (DF) is the function *f* such that

$$f(\mathbf{x}, \mathbf{v}, t)\,\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{v}$$

  is the probability that at time *t* a randomly chosen star has phase-space coordinates in the given range
- The DF is normalized to unity via the integral over the whole phase space

$$\int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{v}\, f(\mathbf{x}, \mathbf{v}, t) = 1$$

Relation to observables

- The probability of finding a star at **x** regardless of velocity is

$$\nu(\mathbf{x}) \equiv \int \mathrm{d}^3\mathbf{v}\, f(\mathbf{x}, \mathbf{v})$$

- If we multiply by the total number of stars *N*, we will obtain the spacial number density of stars

$$n(\mathbf{x}) \equiv N\nu(\mathbf{x})$$

- If we normalize the distribution function by the total mass *M* or luminosity *L*, the density will describe the mass density or luminosity density

Relation to observables

- The probability distribution of stellar velocities at **x** is given by

$$P_{\mathbf{x}}(\mathbf{v}) = \frac{f(\mathbf{x},\mathbf{v})}{\nu(\mathbf{x})}$$

- The mean velocity of stars at **x** is

$$\overline{\mathbf{v}}(\mathbf{x}) \equiv \int \mathrm{d}^3\mathbf{v}\,\mathbf{v}\,P_{\mathbf{x}}(\mathbf{v}) = \frac{1}{\nu(\mathbf{x})}\int \mathrm{d}^3\mathbf{v}\,\mathbf{v}\,f(\mathbf{x},\mathbf{v}).$$

- The velocity dispersion tensor is

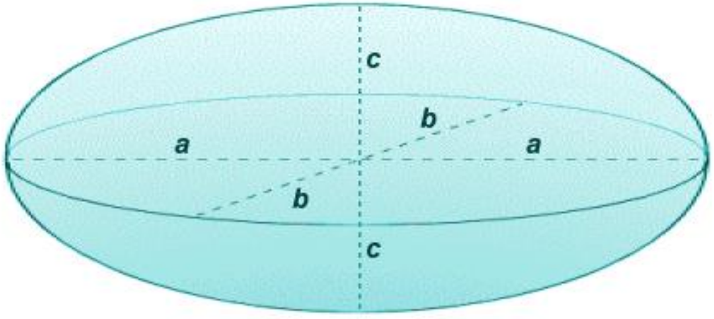
$$\sigma^2_{ij}(\mathbf{x}) \equiv \frac{1}{\nu(\mathbf{x})}\int \mathrm{d}^3\mathbf{v}\,(v_i - \overline{v}_i)(v_j - \overline{v}_j)f(\mathbf{x},\mathbf{v})$$
$$= \overline{v_i v_j} - \overline{v}_i\overline{v}_j\,.$$

The velocity ellipsoid

- The tensor is symmetric so can be diagonalized to the form

$$\sigma_{ij}^2 = \sigma_{ii}^2 \delta_{ij}$$

- The velocity ellipsoid is the ellipsoid with principal axes oriented along the diagonalizing coordinates and semi-axis lengths given by a=σ_{11}, b=σ_{22} and c=σ_{33}



Boltzmann equation

- The requirement that the probability must be conserved leads to an analog of the continuity equation which in the Carthesian coordinates can be written in the form

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

- Another way to express the meaning of this equation is to say that the flow of the probability fluid through phase space is incompressible

Integrals of motion

- The integral of motion satisfies the following condition along the orbit

$$\frac{\mathrm{d}}{\mathrm{d}t} I[\mathbf{x}(t), \mathbf{v}(t)] = 0$$

- This equation can be rewritten as

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial \mathbf{x}} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + \frac{\partial I}{\partial \mathbf{v}} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 0 \quad \text{or} \quad \mathbf{v} \cdot \frac{\partial I}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \frac{\partial I}{\partial \mathbf{v}} = 0$$

- So the condition for *I* to be an integral is the same as to be the steady-state solution of the Boltzman equation

Jeans theorem

- Any steady-state solution of the collisionless Boltzmann equation depends on the phase-space coordinates only through integrals of motion in the given potential
- Any function of the integrals yields a steady-state solution of the collisionless Boltzmann equation

The second part of the theorem is especially useful because it allows us to construct DFs

DFs depending on energy

- In any steady-state potential the Hamiltonian *H* is an integral of motion
- We may construct distributions functions as any non-negative functions of *H*
- DFs of this form are called ergodic
- An isolated system with ergodic DF is spherical
- For potentials that are constant in an inertial frame

$$H = \tfrac{1}{2}v^2 + \Phi(\mathbf{x})$$

- In this case the mean velocity vanishes everywhere because the integrand is an odd function of velocity

$$\overline{\mathbf{v}}(\mathbf{x}) = \frac{1}{\nu(\mathbf{x})}\int \mathrm{d}^3\mathbf{v}\,\mathbf{v}\,f(\tfrac{1}{2}v^2 + \Phi) = 0$$

DFs depending on energy

- In addition to the absence of streaming motions, the velocity dispersion tensor is isotropic

$$\sigma_{ij}^2 = \overline{v_i v_j} = \sigma^2 \delta_{ij} \qquad \text{with}$$

$$\begin{aligned}
\sigma^2(\mathbf{x}) &= \frac{1}{\nu(\mathbf{x})} \int \mathrm{d}v_z \, v_z^2 \int \mathrm{d}v_x \mathrm{d}v_y \, f[\tfrac{1}{2}(v_x^2 + v_y^2 + v_z^2) + \Phi(\mathbf{x})] \\
&= \frac{4\pi}{3\nu(\mathbf{x})} \int_0^\infty \mathrm{d}v \, v^4 f(\tfrac{1}{2}v^2 + \Phi).
\end{aligned}$$

- So the velocity ellipsoid is a sphere of radius σ
- Every system with an ergodic DF is isotropic (has isotropic velocity dispersion tensor)

DFs depending on E and L

- In a spherical potential the angular momentum vector is also an integral of motion
- Usually DFs for spherical systems are constructed using the energy and the value of L: $f = f(H, L)$
- It is then useful to define the radial velocity (parallel to the radial direction) v_r and the tangential velocity (perpendicular) $v_t^2 = v_\theta^2 + v_\phi^2$
- Then the angular momentum $L=r\, v_t$ and $H=(v_r^2+v_t^2)/2 + \Phi(r)$
- The mean velocities vanish but the system does not have to be isotropic: $\sigma_\theta^2 = \sigma_\phi^2 \neq \sigma_r^2$ because the DF depends differently on v_r and v_t

DFs depending on E and L_z

- In an axisymmetric potential the angular momentum component L_z is an integral of motion
- DFs od such systems are constructed using the energy and L_z : $f = f(H, L_z)$
- In cylindrical coordinates $L_z = R\, v_\phi$ and

  $H=(v_R^2+v_z^2+v_\phi^2)/2 + \Phi$
- The mean velocities along R and z vanish, but are non-zero along ϕ
- The system is in general not isotropic $\sigma_R^2=\sigma_z^2\neq\sigma_\phi^2$
- Models with $\sigma_R^2 = \sigma_z^2 = \sigma_\phi^2$ are called isotropic rotators

Eddington's formula

- For a spherical stellar system with a known potential it is possible to derive a unique ergodic DF that depends only on energy

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\left[\int_0^{\mathcal{E}} \frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E}-\Psi}}\frac{\mathrm{d}^2\nu}{\mathrm{d}\Psi^2} + \frac{1}{\sqrt{\mathcal{E}}}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\Psi}\right)_{\Psi=0}\right]$$

with the relative energy and potential

$$\Psi \equiv -\Phi + \Phi_0 \quad \text{and} \quad \mathcal{E} \equiv -H + \Phi_0 = \Psi - \tfrac{1}{2}v^2$$

- Given a spherical density distribution, we can recover an ergodic DF that generates a model with the given density
- There is no guaratee that f will be physical, $f \geq 0$

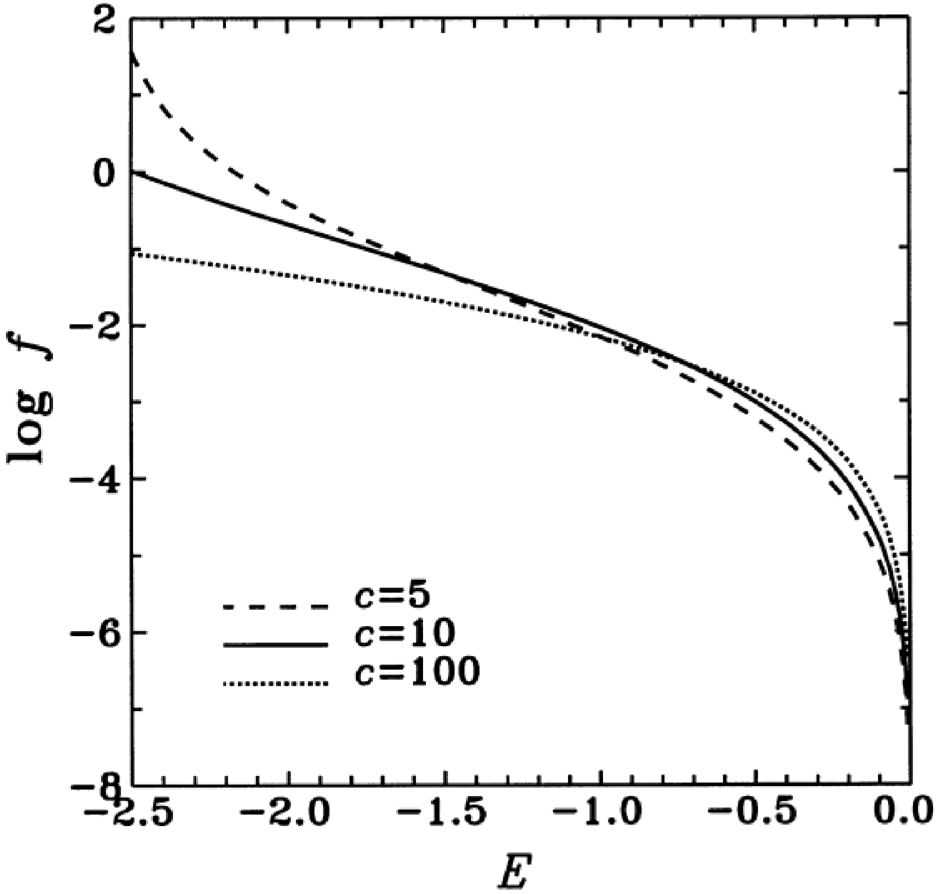
Examples of DFs

- Analytic expressions for DFs generated by some popular density profiles are available
- One example is the DF for the Hernquist profile

$$f_{\rm H}(\mathcal{E}) = \frac{1}{\sqrt{2}(2\pi)^3(GMa)^{3/2}} \frac{\sqrt{\widetilde{\mathcal{E}}}}{(1-\widetilde{\mathcal{E}})^2} \times \left[(1-2\widetilde{\mathcal{E}})(8\widetilde{\mathcal{E}}^2 - 8\widetilde{\mathcal{E}} - 3) + \frac{3\sin^{-1}\sqrt{\widetilde{\mathcal{E}}}}{\sqrt{\widetilde{\mathcal{E}}(1-\widetilde{\mathcal{E}})}}\right]$$

$$\widetilde{\mathcal{E}} \equiv -Ea/GM$$

Examples of DFs



- For NFW profile no analytical formula is available and the DF has to be calculated numerically
- The dependence on energy is similar as for other well-known density profiles

Łokas & Mamon 2001
Widrow 2000

Models with constant anisotropy

- Among models dependent on *E* and *L* a particularly simple and useful case is that with constant anisotropy

$$f(\mathcal{E}, L) = L^{-2\beta} f_1(\mathcal{E})$$

- The anisotropy parameter $\beta = 1 - (\sigma_\theta^2 + \sigma_\phi^2) / (2\, \sigma_r^2)$ describes the type of stellar orbits in the system:

  $\beta = 0$ – isotropic orbits

  $\beta = 1$ – radial orbits ($\sigma_\theta = \sigma_\phi = 0$)

  $\beta = -\infty$ – circular orbits ($\sigma_r = 0$)

Osipkov-Merritt models

- In simulated objects like galaxies and galaxy clusters we often find the anisotropy to increase with radius
- A model that reproduces this trend to some extent is the Osipkov-Merritt model with anisotropy

$$\beta(r) = \frac{1}{1 + r_{\rm a}^2/r^2}$$

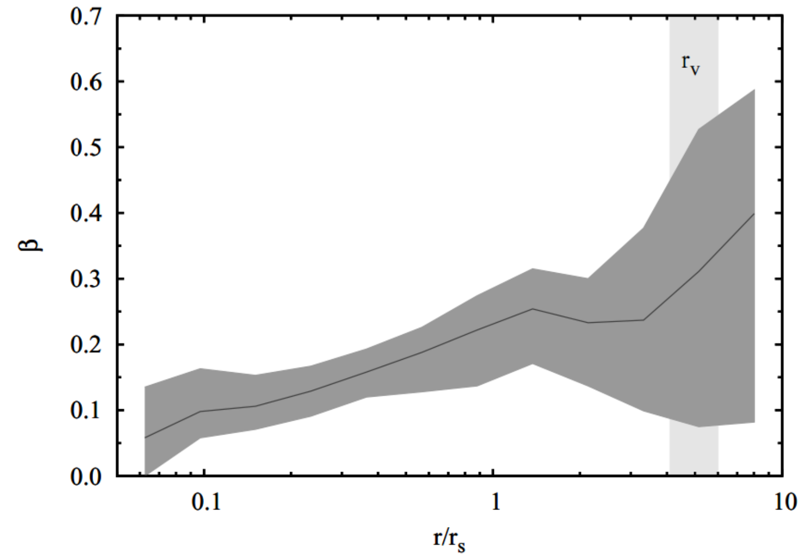
where r_a is the anisotropy radius

- The DF of this model depends on energy and angular momentum only via the variable

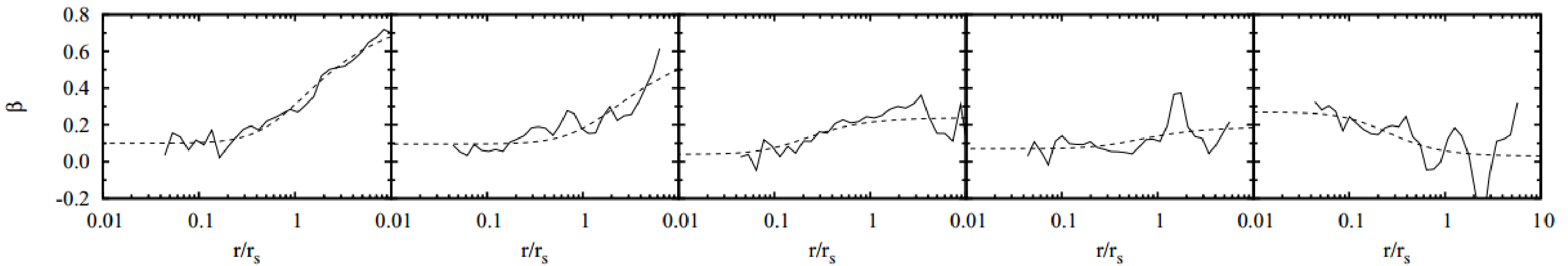
$$Q \equiv \mathcal{E} - \frac{L^2}{2r_{\rm a}^2}$$

Anisotropy profiles for DM haloes

- On average, the simulated DM haloes have increasing anisotropy profiles, isotropic in the center, radially biased outside



- Some haloes show decreasing profiles



Wojtak et al. 2008

More general DFs

- A more general formula that encompasses all these possibilities is

$$f(E,L) = f_E(E) f_L(L)$$

- The angular momentum part is given by

$$f_L(L) = L^{-2\beta_0} \left(1 + \frac{L^2}{2L_0^2}\right)^{\beta_0 - \beta_\infty}$$

- The anisotropy parameter has the limiting values β_0 in the center and β_∞ at infinity

Spherical systems

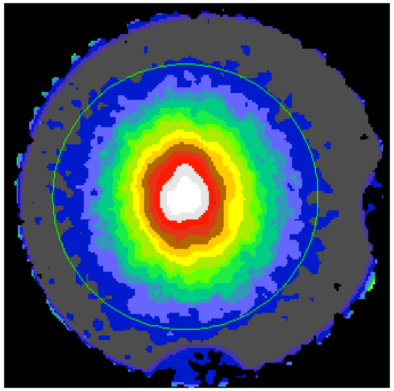
- At first approximation many stellar systems may be treated as spherical
- These include: globular clusters, dwarf spheroidals, elliptical galaxies and galaxy clusters
- The models can be applied to systems of mass in the range 10^5-10^{15} $M_\odot$

GC 47 Tuc

dSph Leo I

Galaxy M87

A3571



Globular clusters

- Most of them are spherical to a very good approximation so spherical models are especially well justified and not subject to any biases
- They are relatively simple to model because they are not expected to contain dark matter so mass-follows-light models can be used
- Many of them can be found in the Milky Way, are relatively close (even at a few kpc) so the spectra and then the velocities of the stars are easy to measure

Dwarf spheroidals

- Less spherical than globular clusters with typical ellipticities of 0.3
- More distant, typically at 20-200 kpc away, so measurements of velocities are more challenging
- Believed to contain a lot of dark matter
- Properties of dark matter cause substantial uncertainties in the modelling and more parameters need to be constrained

Elliptical galaxies

- Planetary nebulae or globular clusters can be used as tracers of the potential
- The tracers are usually scarce at distances larger than the effective radius so it is difficult to constrain the distribution of dark matter
- Many probably contain black holes that influence the velocity distribution of the stars and make the modelling more complicated

Galaxy clusters

- Biggest gravitationally bound structures: may not yet be in equilibrium and may be affected by infalling material
- Departures from sphericity are difficult to measure
- Believed to be strongly dominated by dark matter and the distribution of galaxies is believed to trace the mass
- Different populations of galaxies are present, ellipticals are more reliable for modelling as they probably spent more time in the cluster while spirals may be infalling for the first time

How to proceed without DF

- Comparisons between theoretical models and observational data are usually performed in terms of velocity moments, such as mean rotation velocity and velocity dispersion
- Calculating these moments from a known DF is easy
- However, finding an appropriate DF is not straightforward
- Even if a DF is found it may not be unique
- It is therefore useful to infer moments without recovering the actual DF

Jeans equation

$$\frac{d(\nu\sigma_r^2)}{dr} + 2\frac{\beta}{r}\nu\sigma_r^2 - \nu\frac{d\Phi}{dr} = 0$$

σ_r - radial velocity dispersion

$\beta = 1 - (\sigma_\theta^2 + \sigma_\phi^2)/(2\sigma_r^2)$ – anisotropy parameter

$\beta = 0$ – isotropic orbits

$\beta = 1$ – radial orbits

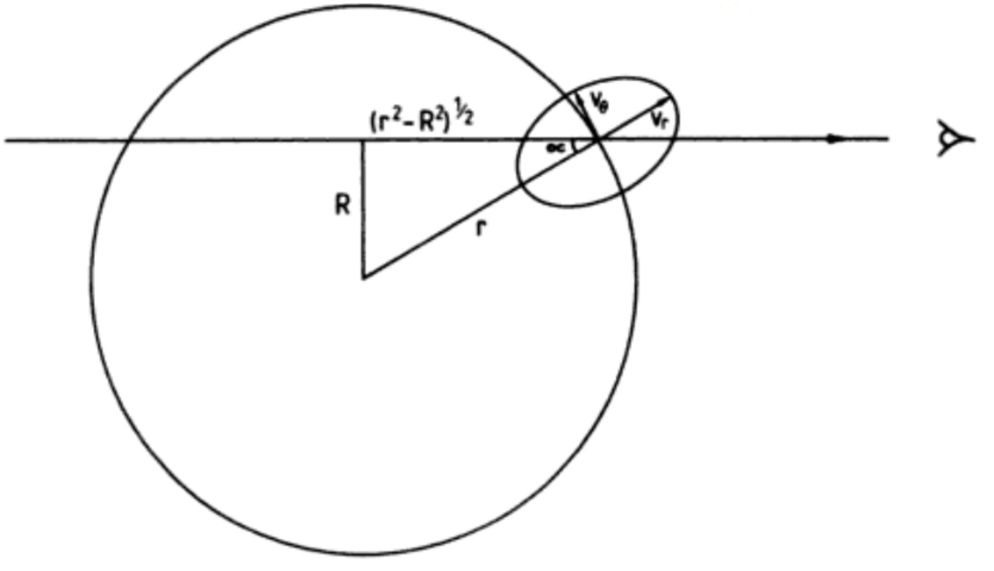
$\beta = -\infty$ – circular orbits

$\nu(r)$ – 3D distribution of stars

$d\Phi/dr$ – gravitational acceleration

Observed velocity dispersion

$$\sigma_{los}^2(R) = \frac{2}{I(R)} \int_R^\infty \left(1 - \beta \frac{R^2}{r^2}\right) \frac{\nu \sigma_r^2 r}{\sqrt{r^2 - R^2}} dr$$



σ_{los} – line-of-sight velocity dispersion

I (R) – surface distribution of the stars

Solutions of the Jeans equation

- For an isotropic system the solution is very simple

$$\nu\sigma_r^2(r)(\beta=0)=\int_r^\infty \nu\frac{d\Phi}{ds}ds$$

- The line-of-sight velocity dispersion is then

$$\sigma_{los}^2(R)(\beta=0)=\frac{2}{I(R)}\int_R^\infty\frac{r}{\sqrt{r^2-R^2}}\int_r^\infty \nu\frac{d\Phi}{ds}ds\,dr$$

$$=\frac{2G}{I(R)}\int_R^\infty\frac{\sqrt{r^2-R^2}}{r^2}\nu(r)M(r)dr$$

- For some density/mass distributions this can even be calculated analytically

Solutions of the Jeans equation

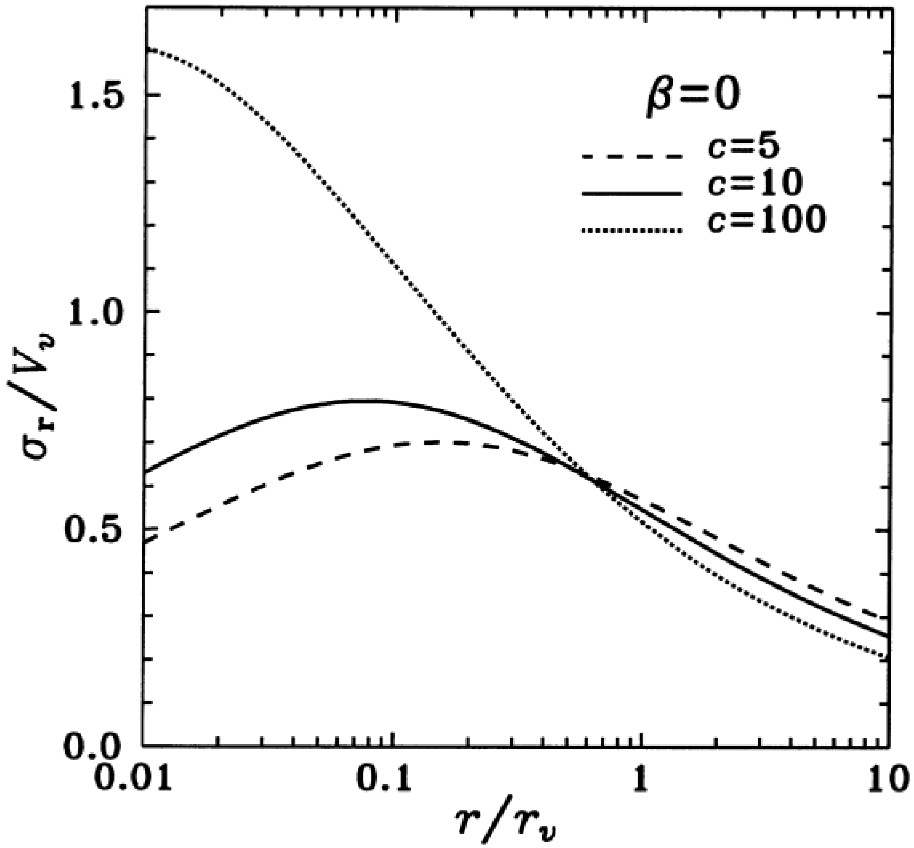
- Another interesting case is the class of systems with a distribution function f = f(E) $L^{-2\beta}$ where the anisotropy parameter is constant
- For example, the solution for the NFW model is then

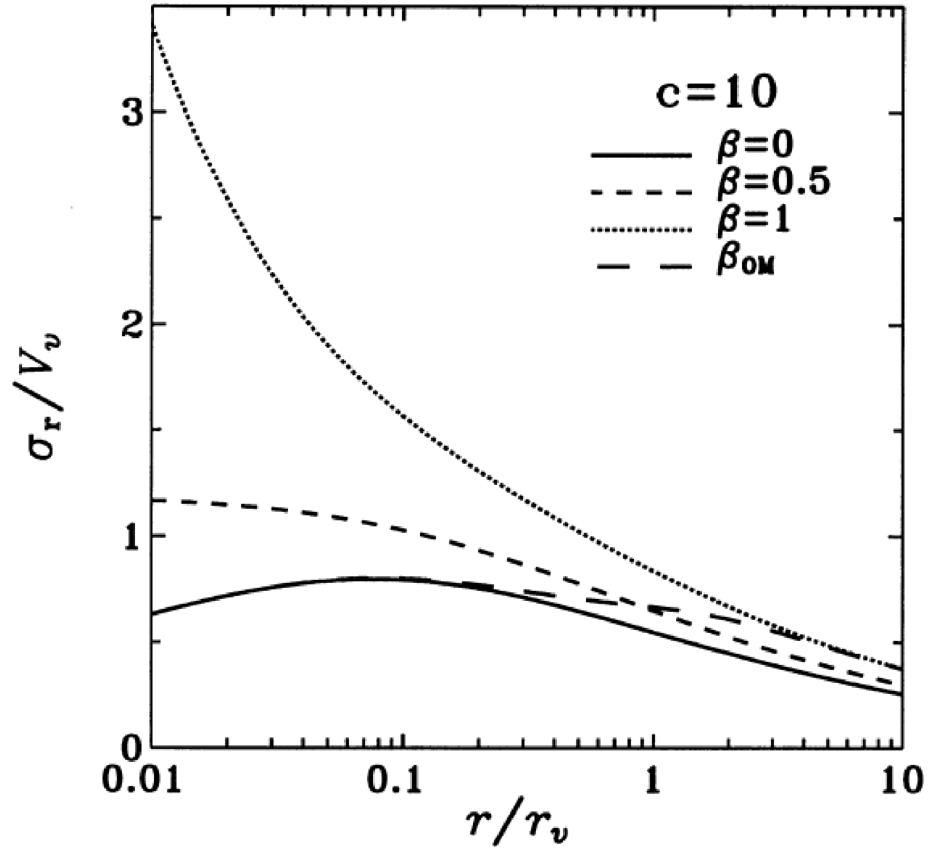
$$\frac{\sigma_{\mathrm{r}}^2}{V_{\mathrm{v}}^2}(s, \beta = \mathrm{const}) = g(c)(1+cs)^2 s^{1-2\beta}$$

$$\times \int_s^\infty \left[\frac{s^{2\beta-3}\ln(1+cs)}{(1+cs)^2} - \frac{cs^{2\beta-2}}{(1+cs)^3}\right] \mathrm{d}s$$

with $s=r/r_v$, c is the concentration parameter and V_v is the circular velocity at the virial radius

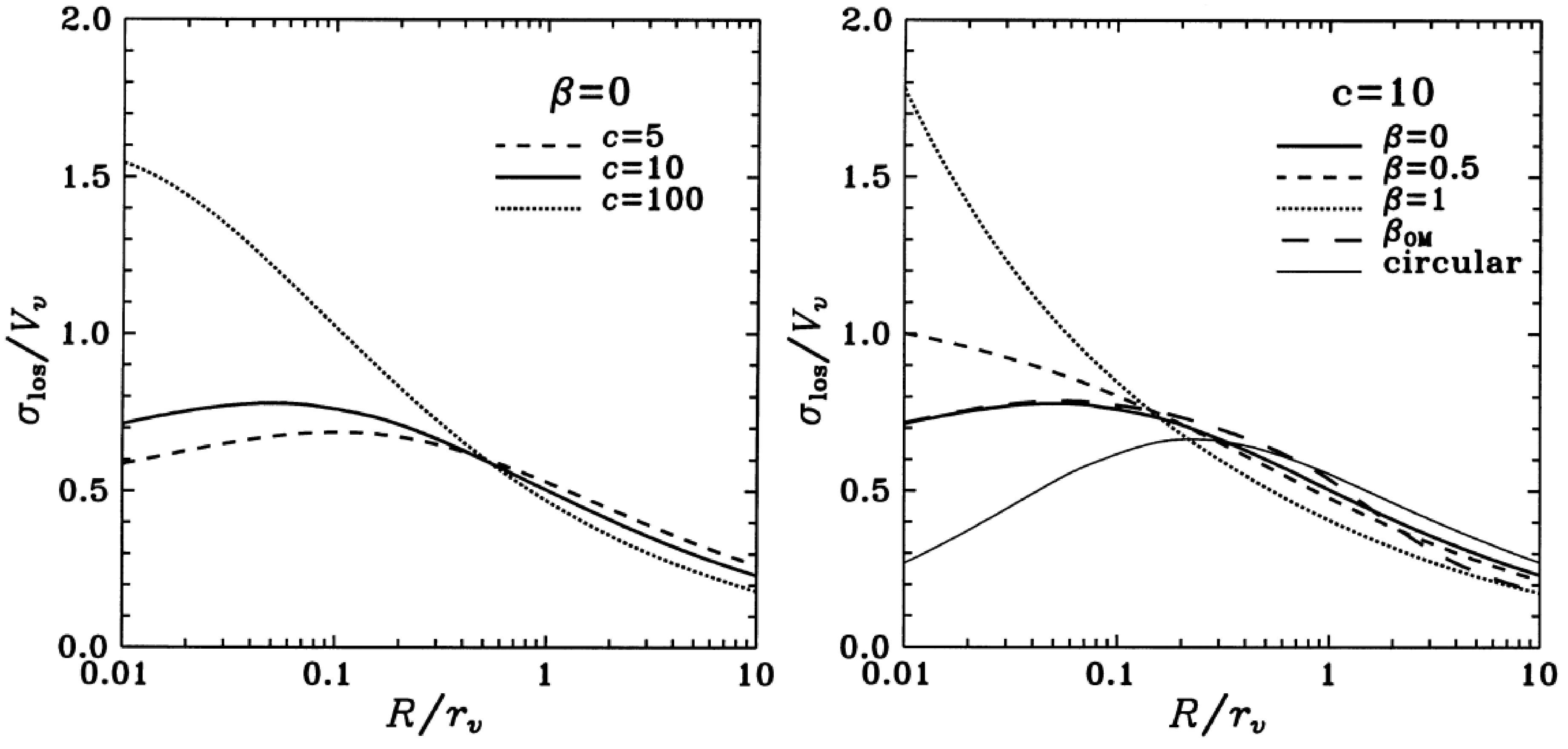
Radial velocity dispersion





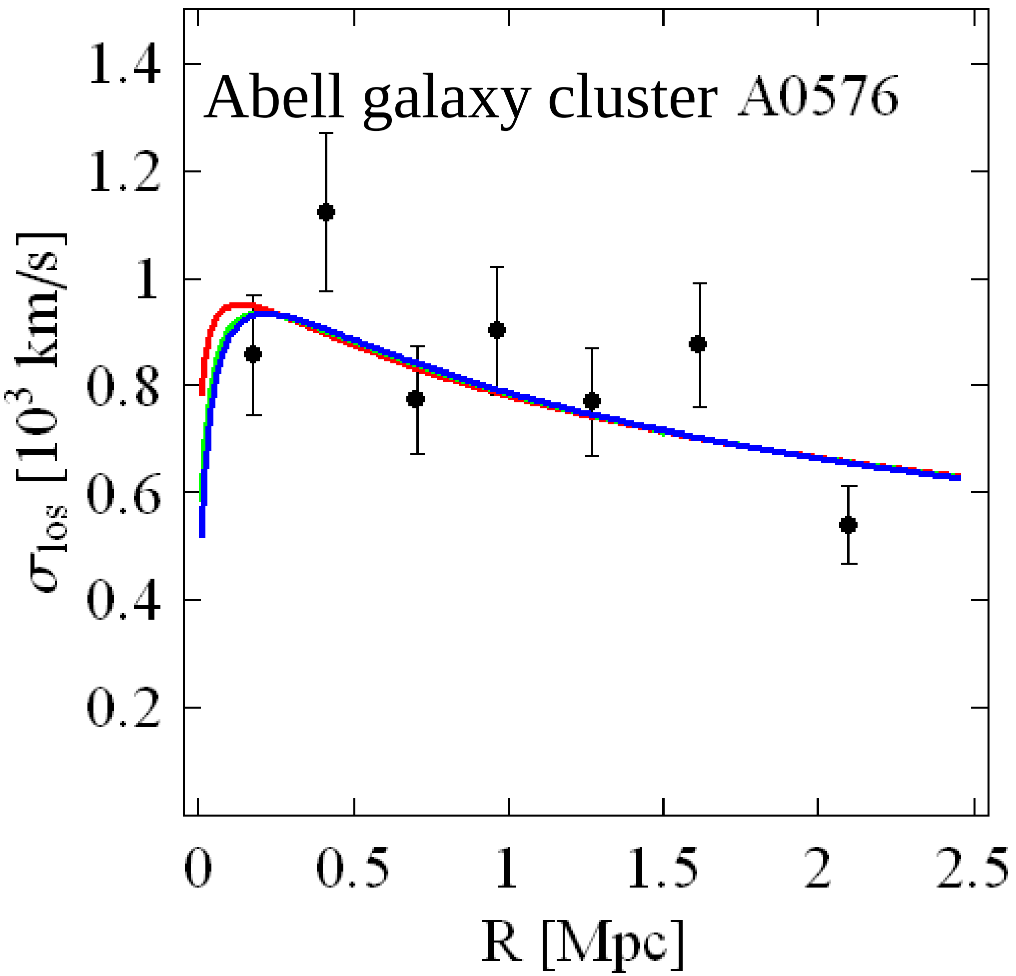
Steeper density profiles for a fixed β have a similar effect as more radial orbits for a fixed profile

LOS velocity dispersion



The same fundamental degeneracy is present for los dispersion

Mass-anisotropy degeneracy



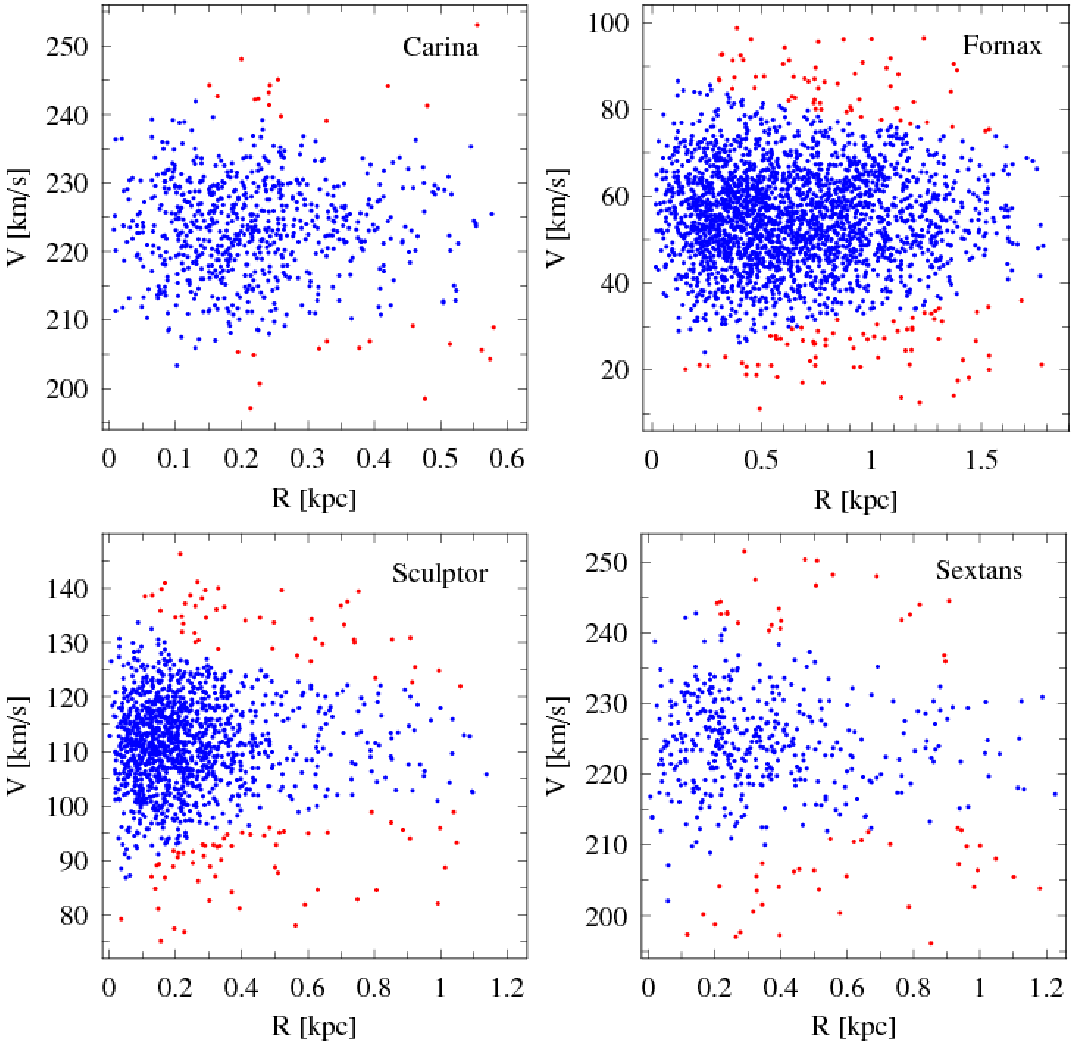
$\beta = 0, \quad c = 9.2$

$\beta = -1, c = 13.6$

$\beta = -2, c = 15.7$

Different combinations of anisotropy and concentration reproduce the data equally well

Examples of data sets



- Kinematic data sets for dSph galaxies from Walker et al. (2008)
- The data include the positions of stars and their LOS velocities
- From these we calculate the velocity dispersion profiles

Measuring velocity dispersion

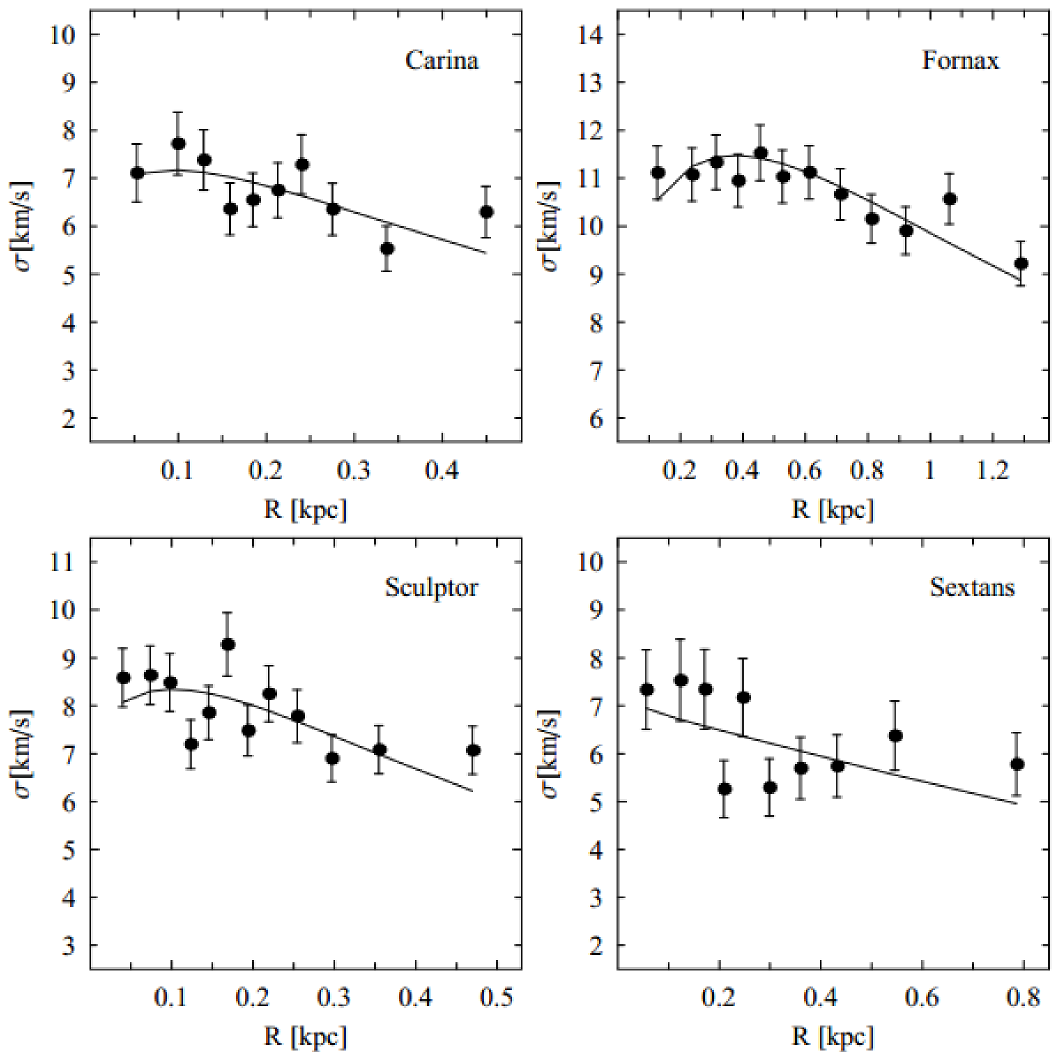
- The velocity dispersion σ_{los} for a sample of *n* stars is measured using the estimator

$$s = \left[\frac{1}{n-1} \sum_{i=1}^{n} (v_i - \bar{v})^2 \right]^{1/2} \quad \text{where} \quad \bar{v} = \frac{1}{n} \sum_{i=1}^{n} v_i$$

- The data can be binned to produce the velocity dispersion profile as a function of the projected radius *R*
- The measurements are assigned sampling errors

$$\Delta s = \frac{s}{\sqrt{2(n-1)}}$$

Fitting the dispersion profiles



- The dispersion profiles are fitted with the solutions of the Jeans equation
- If we assume that mass follows light we may find the best-fitting parameters such as M/L and β

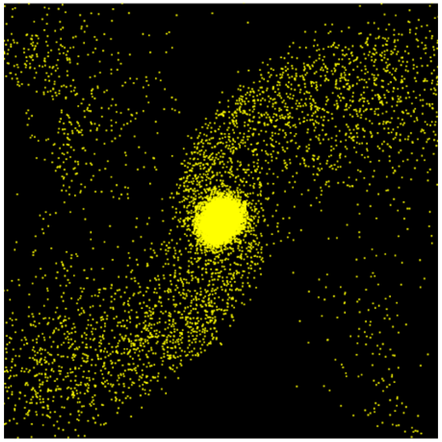
Łokas 2009

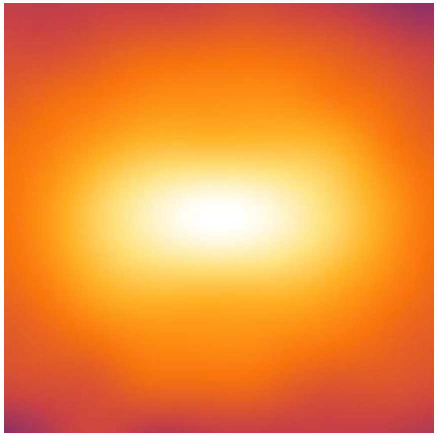
Fitted parameters

dSph galaxy	Mass [10^7 $M_\odot$]	β	M/L_V [$M_\odot/L_\odot$]
Carina	2.3 ± 0.2	-0.04	67 ± 31
Fornax	15.7 ± 0.7	-0.33	8.8 ± 3.8
Sculptor	3.1 ± 0.2	-0.09	15.3 ± 6.9
Sextans	4.0 ± 0.6	0.22	91 ± 49

Problems in modelling dSphs

- dSphs may be affected by tidal forces from the Milky Way
- If they formed from disks they may be significantly non-spherical
- They may have some remnant rotation
- The first two effects may bias the results toward higher masses and/or tangential orbits

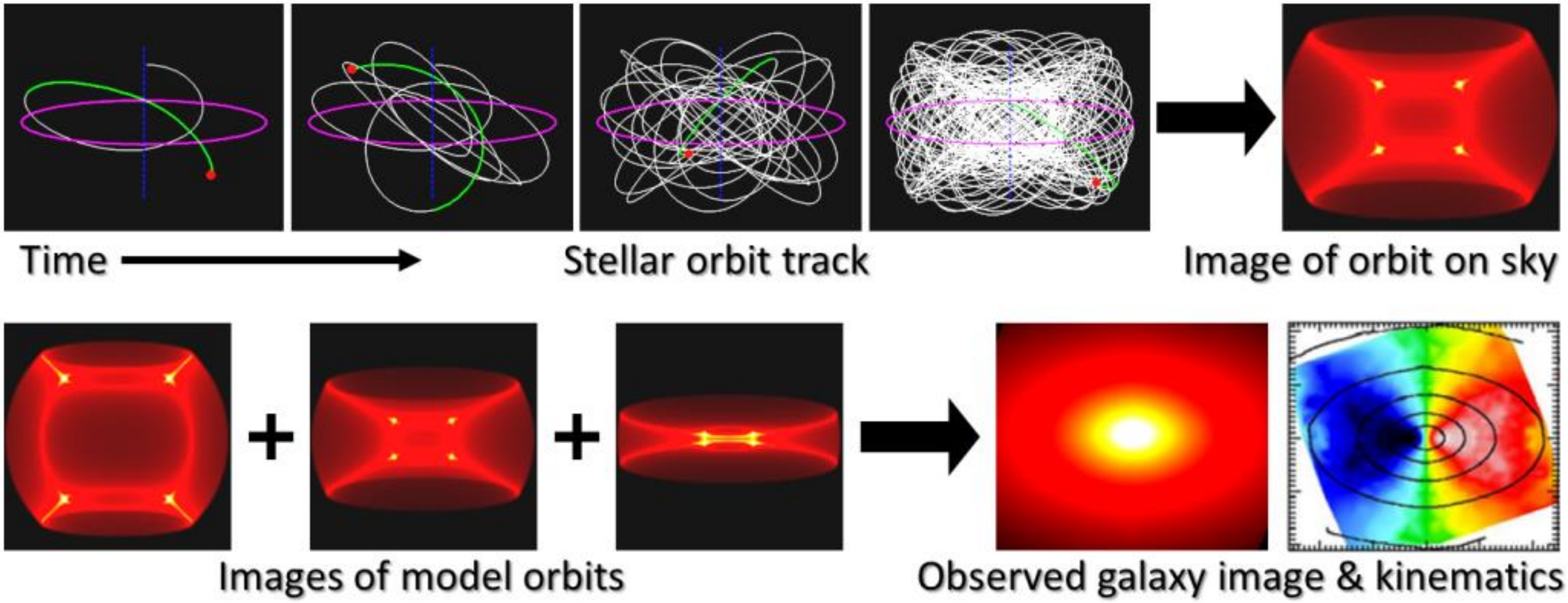




Schwarzschild's method

- Schwarzschild's approach is intermediate between the analytic techniques based on DFs and N-body simulations which follow the evolution of individual particles (particle-based)
- Schwarzschild's method combines orbits to create a stellar system and hence is called an orbit-based method

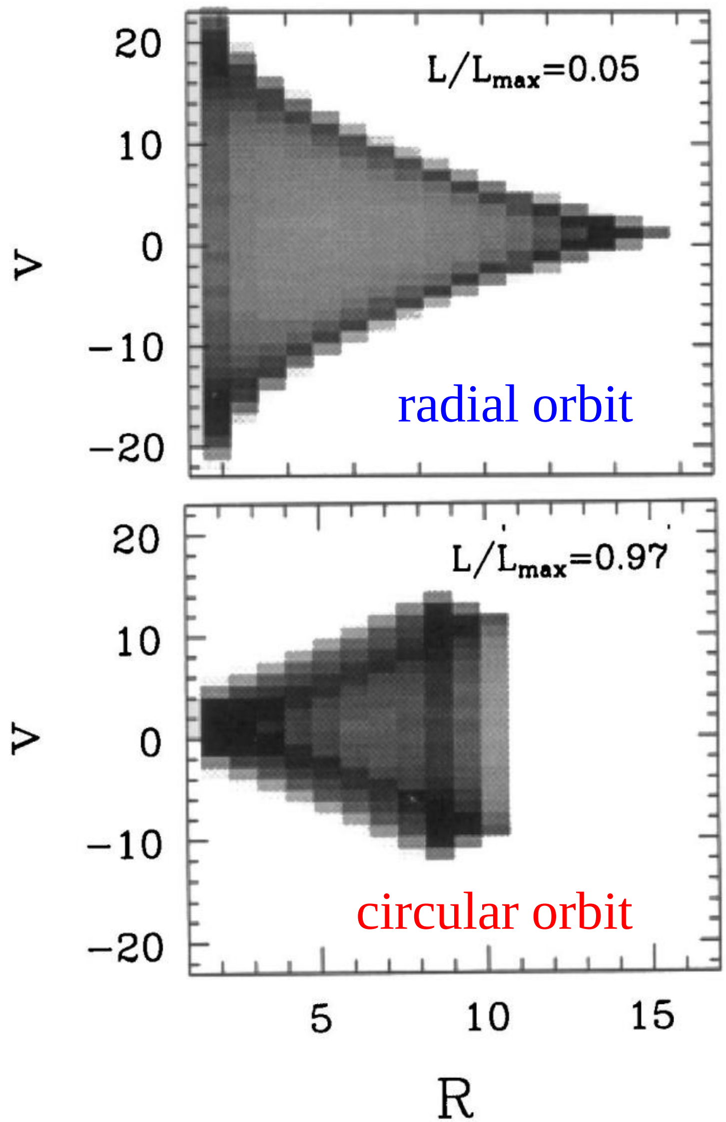
Orbit-based models



We create a library of orbits and find their relative contributions to the observed density and velocity distribution of stars in a galaxy

Cappellari 2014

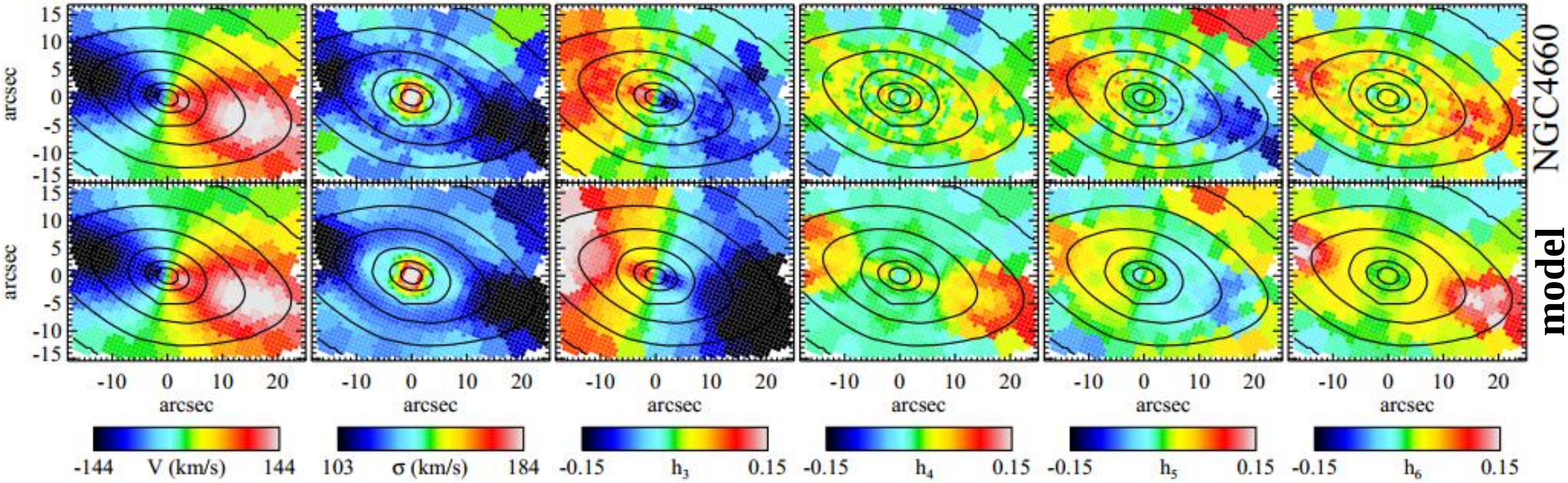
Contribution of different orbits



- Radial orbits contribute to the velocity distribution in a different way than circular orbits
- This allows us to construct models that reproduce the data best
- The models are not necessarily unique

Rix et al. 1997

Orbit-based models



The kinematics is fitted by reproducing the velocity distribution of stars in terms of different velocity moments

Cappellari 2014