Radiative Processes

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Contents

1 Electromagnetic radiation

Wave-particle duality. Electromagnetic radiation can be described *classically* as *electromagnetic waves* or *quantum-mechanically* as a collection of *photons*.

Electromagnetic wave. Maxwell's equations in Gaussian cgs units:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho_e; \qquad \vec{\nabla}\cdot\vec{B} = 0; \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}; \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}. \tag{1.1}$$

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, ρ_e is the electric charge density, $\vec{j}$ is the electric current density.

In vacuum ($\rho_e = 0$ and $\vec{j} = 0$), those equations become symmetric:

$$\vec{\nabla}\cdot\vec{E} = 0; \qquad \vec{\nabla}\cdot\vec{B} = 0; \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}; \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}. \tag{1.2}$$

Their symmetry allows to obtain identical wave equations:

$$\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \nabla^2\vec{E}; \qquad \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = \nabla^2\vec{B}. \tag{1.3}$$

A solution of the form $\vec{E} = \vec{E}_0\exp(i\omega t + i\vec{k}\cdot\vec{r})$ with angular (ordinary) frequency $\omega = 2\pi\nu$ and wavevector $\vec{k}$ yields the dispersion relation $\omega^2 = k^2c^2$ or wavelength $\lambda = 2\pi/k = 2\pi c/\omega = c/\nu$, and ties the amplitude vectors with $\hat{k} = \vec{k}/k$: $\vec{B}_0 = \vec{E}_0\times\hat{k}$ and $\vec{E}_0 = -\vec{B}_0\times\hat{k}$, hence $\vec{E}\perp\vec{B}$ and $E_0 = B_0$.

Photons. Photons are massless particles propagating at constant speed $c = \lambda\nu$ — *the speed of light in vacuum*[1]. A photon can be described by momentum $\vec{p} = p\hat{k}$, which corresponds to energy $\epsilon = pc$, which is related to the wave frequency as $\epsilon = h\nu = \hbar\omega$, with $h = 2\pi\hbar$ — *the Planck constant*[2].

The parameters λ, ν or ω, and ϵ describe the *electromagnetic spectrum*, from radio waves to gamma rays.

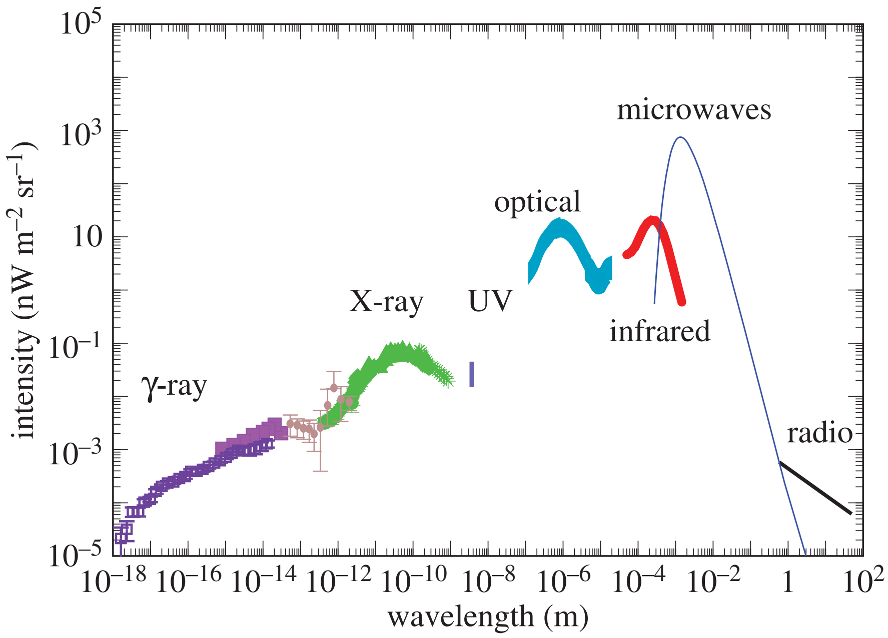

Figure 1: Specific intensity of the cosmic radiation background (Cooray 2016, Royal Society Open Science, 3, 150555).

2 Radiation measures

Treated as a collection of electromagnetic waves, radiation can be measured by integrating their energy and momentum densities. At any fixed point $\vec{r}$, the time-averaged energy density of electromagnetic

[1] $c = 2.99792458\times10^{10}$cm/s
[2] $h = 6.62607015\times10^{-27}$erg/s

wave is

$$u = u_{\mathrm{E}} + u_{\mathrm{B}} = \frac{\langle E^2 \rangle + \langle B^2 \rangle}{8\pi} = \frac{E_0^2/2 + B_0^2/2}{8\pi} = \frac{E_0^2}{8\pi}\,, \tag{2.1}$$

and the time-averaged energy flux (momentum times c) density (Poynting flux) is

$$\vec{S} = \frac{c}{4\pi}\left\langle \vec{E} \times \vec{B} \right\rangle = \frac{c}{4\pi}\frac{E_0 B_0}{2}\hat{k} = c\frac{E_0^2}{8\pi}\hat{k}\,. \tag{2.2}$$

Treated as a collection of photons, radiation can be measured by counting the photons number N or by integrating their combined energy $\mathcal{E} = \sum_{i=1,N} \epsilon_i$.

Phase space element. A photon can be located in the 6-dimensional phase space $(\vec{r}, \vec{p})$. The element of position space (volume) can be decomposed as $\mathrm{d}V = \mathrm{d}^3\vec{r} = \mathrm{d}A_\perp\, c\mathrm{d}t$ with $\mathrm{d}A_\perp$ the area element perpendicular to the propagation direction $\hat{k}$. The element of momentum space can be decomposed as

$$\mathrm{d}^3\vec{p} = p^2\,\mathrm{d}p\,\mathrm{d}\Omega = \frac{\epsilon^2}{c^3}\,\mathrm{d}\epsilon\,\mathrm{d}\Omega = \frac{h^3\nu^2}{c^3}\,\mathrm{d}\nu\,\mathrm{d}\Omega\,, \tag{2.3}$$

with $\mathrm{d}\Omega$ the solid angle element[3]. Hence, the element of phase space can be written as $\mathrm{d}^3\vec{r}\,\mathrm{d}^3\vec{p} = (h^3\nu^2/c^2)\,\mathrm{d}A_\perp\,\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega$.

Specific intensity. Consider a measure of radiation energy per phase space element:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}^3\vec{r}\,\mathrm{d}^3\vec{p}} = \frac{c^2}{h^3\nu^2}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}A_\perp\,\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega} \equiv \frac{c^2}{h^3\nu^2}I_\nu \tag{2.4}$$

where $I_\nu \equiv \mathrm{d}\mathcal{E}/(\mathrm{d}A_\perp\,\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega)$ is the *specific intensity*, a fundamental radiation measure in astrophysics, from which other measures can be obtained by integration.

Total intensity. The *total (bolometric) intensity* is the integral of specific intensity over all frequencies $I \equiv \int \mathrm{d}\nu\, I_\nu = \mathrm{d}\mathcal{E}/(\mathrm{d}A_\perp\,\mathrm{d}t\,\mathrm{d}\Omega)$. Other measures can also be defined in the specific or total versions.

Lorentz invariance. A phase space element is invariant to Lorentz transformation. For a boost along x by Lorentz factor Γ, the length contraction $\mathrm{d}x = \mathrm{d}x'/\Gamma$ is compensated by the momentum boost $\mathrm{d}p_x = \Gamma\,\mathrm{d}p'_x$. Since $\mathrm{d}\mathcal{E} = \epsilon\,\mathrm{d}N = h\nu\,\mathrm{d}N$, and the number of photons $\mathrm{d}N$ is invariant, I_ν/ν^3 is also invariant.

Energy density. The *energy density* is a measure of radiation energy per volume element:

$$u \equiv \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}V} = \frac{1}{c}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}A_\perp\,\mathrm{d}t}\,. \tag{2.5}$$

It is thus equivalent to the intensity $I(\Omega)$ integrated over all solid angles: $u = \int_{4\pi} \mathrm{d}\Omega\, I(\Omega)/c$.

Energy flux (density). The *energy flux density* is a measure of radiation energy crossing a surface element (in particular of a source or a detector) in unit time:

$$F \equiv \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}A\,\mathrm{d}t}\,. \tag{2.6}$$

Such radiation may consist of various beams of intensity $I(\Omega)$ making angles θ with the unit vector $\hat{n}$ normal to the surface element. For each such beam, the volume occupied by photons that cross the surface over time range $\mathrm{d}t$ is $\mathrm{d}V(\Omega) = c\,\mathrm{d}t\,\mathrm{d}A\,\cos\theta$. Noting that $\mathrm{d}\mathcal{E}(\Omega) = I(\Omega)\mathrm{d}V(\Omega)\,\mathrm{d}\Omega/c$, one finds:

$$F = \int_{4\pi} \mathrm{d}\Omega \frac{\mathrm{d}\mathcal{E}(\Omega)}{\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\Omega} = \int_{4\pi} \mathrm{d}\Omega \frac{I(\Omega)\,\mathrm{d}V(\Omega)/c}{\mathrm{d}A\,\mathrm{d}t} = \int_{4\pi} \mathrm{d}\Omega\, I(\Omega)\cos\theta\,. \tag{2.7}$$

[3]For example, in spherical coordinates (r, θ, ϕ) a solid angle element anchored at $r = 0$ is $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \mathrm{d}\mu\,\mathrm{d}\phi$, where $\mu = \cos\theta$.

Pressure. Let $\mathcal{P}_\perp = \sum_i p_{\perp,i} = \sum_i \epsilon_i \cos\theta_i / c$ be the combined momentum of radiation normal to a given surface. One can define the *momentum flux (density)* as

$$F_p \equiv \frac{\mathrm{d}\mathcal{P}_\perp}{\mathrm{d}A\,\mathrm{d}t} = \int_{4\pi} \mathrm{d}\Omega\, I(\Omega)\cos^2\theta/c\,. \tag{2.8}$$

The momentum flux density is closely related to the *radiation pressure* $P_{\rm rad}$ exerted on a medium, depending on how they interact. If the radiation is completely absorbed, $P_{\rm rad} = F_p$; if the radiation is completely reflected, $P_{\rm rad} = 2F_p$.

Isotropic radiation. Let $I(\Omega) = I$. The energy density is $u = 4\pi I/c$, the energy flux density is $F = 0$, and the momentum flux density is $F_p = 4\pi I/3c = u/3$. One can also calculate the *emergent energy flux density* (one-way flux, i.e., $\mu = \cos\theta > 0$): $F_+ = \int_{2\pi} \mathrm{d}\Omega\, I\cos\theta = 2\pi I \int_0^1 \mathrm{d}\mu\,\mu = \pi I$.

Unidirectional radiation. Let $I(\Omega) = I\,\delta(\theta)/(2\pi\sin\theta)$, so that $u = (I/2\pi c)\int \mathrm{d}\theta\,\mathrm{d}\phi\,\delta(\theta) = I/c$. The energy flux density is $F = I\int_0^\pi \mathrm{d}\theta\,\delta(\theta)\cos\theta = I$, and the momentum flux density is $F_p = (I/c)\int_0^\pi \mathrm{d}\theta\,\delta(\theta)\cos^2\theta = I/c$. Hence, in this case $u = F/c = F_p = I/c$.

Luminosity. The *luminosity* (energy flux, power) is a measure of radiation energy emitted by a source in unit time: $L \equiv \mathrm{d}\mathcal{E}/\mathrm{d}t$. Luminosity can be calculated by integrating the emitted energy flux over the source surface: $L = \int \mathrm{d}A\,F(A)$. For a spherical source of radius R and uniform emission flux: $L = 4\pi R^2 F$. In such case, the flux measured at arbitrary distance $r > R$ is $F(r) = L/(4\pi r^2)$ (*inverse square law*).

True luminosities of astrophysical sources are observationally inaccessible. The best one can do is to estimate the *apparent luminosity* $L_{\rm app} = 4\pi d_{\rm L}^2 F_{\rm obs}$ from the observed energy flux density $F_{\rm obs}$ and the luminosity distance $d_{\rm L}$.

3 Spectrum of electromagnetic wave

The spectral distribution of radiation measures can be readily explained in terms of counting photons of various energies. When unidirectional radiation is defined in terms of an electromagnetic wave, its instantaneous intensity (energy flux density, Poynting flux density) is

$$I(t) \equiv F(t) \equiv S(t) \equiv \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t\,\mathrm{d}A_\perp} = \frac{c}{4\pi}E^2(t)\,. \tag{3.1}$$

The electric field $E(t)$, measured for an extended period of time, can be Fourier transformed into the frequency space, e.g.:

$$\hat{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t\,\exp(i\omega t)\,E(t)\,. \tag{3.2}$$

The total energy of such radiation per unit area can be matched between the spaces of time and freuqency:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}A_\perp} = \frac{c}{4\pi}\int_{-\infty}^{\infty} \mathrm{d}t\,E^2(t) = \frac{c}{2}\int_{-\infty}^{\infty} \mathrm{d}\omega\,|\hat{E}(\omega)|^2 = c\int_0^{\infty} \mathrm{d}\omega\,|\hat{E}(\omega)|^2\,. \tag{3.3}$$

The middle transition is due to the *Parseval's theorem*, and the last transition is due to elimination by symmetry of negative frequencies (since $\hat{E}$ is complex). The spectrum of radiation energy deposited over time per unit area is thus:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\omega\,\mathrm{d}A_\perp} = c|\hat{E}(\omega)|^2\,, \qquad \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\nu\,\mathrm{d}A_\perp} = 2\pi c|\hat{E}(\nu)|^2\,. \tag{3.4}$$

4 Polarization

The dispersion relation for electromagnetic wave in vacuum allows two solutions: $\omega = \pm kc$. At any fixed position, e.g. $\vec{r} = 0$, one can write the general complex solution as (allowing for different phases at $t = 0$):

$$\vec{E} = \vec{E}_1\exp(ikct + i\phi_1) + \vec{E}_2\exp(-ikct - i\phi_2)\,. \tag{4.1}$$

The measured electric field is the real part:

$$\mathrm{Re}(\vec{E}) = \vec{E}_1 \cos(kct + \phi_1) + \vec{E}_2 \cos(kct + \phi_2) \tag{4.2}$$

What this describes in general is an ellipse of arbitrary orientation. This ellipse can be traced in two directions (from the observer point of view): clockwise (right-handed polarization; positive helicity) or counterclockwise (left-handed polarization; negative helicity). An ellipse may degenerate to a line (linear polarization) or to a circle (circular polarization).

Stokes parameters. Without losing generality, an observer may choose a coordinate system (x, y) and re-define the electric field amplitudes such that $\vec{E}_1 = E_1\hat{x}$ and $\vec{E}_2 = E_2\hat{y}$. Then, a coherent radiation signal can be determined by measuring 4 scalar parameters: E_1, E_2, ϕ_1, ϕ_2. Those can be used to define the 4 Stokes parameters:

$$\begin{aligned} I &= E_1^2 + E_2^2 && (4.3)\\ Q &= E_1^2 - E_2^2 && (4.4)\\ U &= 2E_1E_2\cos(\phi_1 - \phi_2) && (4.5)\\ V &= 2E_1E_2\sin(\phi_1 - \phi_2) && (4.6) \end{aligned}$$

On the other hand, an arbitrary ellipse can be parametrized in terms of major axis amplitude $E_1' > 0$, minor axis amplitude E_2' ($|E_2'| < E_1'$) and position angle of the major axis χ. The Stokes parameters become:

$$\begin{aligned} I &= E_1'^2 + E_2'^2 && (4.7)\\ Q &= (E_1'^2 - E_2'^2)\cos(2\chi) && (4.8)\\ U &= (E_1'^2 - E_2'^2)\sin(2\chi) && (4.9)\\ V &= 2E_1'E_2' && (4.10) \end{aligned}$$

Linear and circular polarization. Note two important limits:

- $E_2' = 0$ ($V = 0$): the ellipse degenerates to a line, this is the case of *linear polarization*.
- $E_2' = \pm E_1'$ ($Q = U = 0$): the ellipse degenerates to a circle, this is the case of *circular polarization*. The sign of E_2' determines the sign of V, hence the sign of helicity).

Hence, I measures the total intensity, $\sqrt{Q^2 + U^2}$ the linearly polarized intensity with the *electric vector polarization angle* (EVPA) $\chi = \mathrm{arctan2}(U, Q)/2$, V the circularly polarized intensity.

Polarization degree. Note that the above Stokes parameters satisfy $I^2 = Q^2 + U^2 + V^2$ (not independent, since $\phi_1, \phi_2 \to \phi_1 - \phi_2$), meaning that a coherent monochromatic radiation signal is completely polarized. A partial polarization always results from incoherence (e.g., extended source, finite bandwidth, imperfect detector). In general, the measured Stokes parameters satisfy $I^2 \geq Q^2 + U^2 + V^2$. One can introduce the *linear polarization degree* $\Pi = \sqrt{Q^2 + U^2}/I$ and the *circular polarization degree* $\Pi_c = V/I$.

Quantum description of radiation polarization. Every photon can be characterized by a positive or negative spin $\pm\hbar$. Also in the classical description, one can show that a polarized electromagnetic wave carries angular momentum of density $\vec{r} \times \vec{S}$, the sign of which depends on the helicity.

5 Thermal radiation

A system in thermal equilibrium emits radiation with characteristic (Planck or 'blackbody') spectrum and intensity $I_{\mathrm{th}}(\nu, T)$, a function of the system temperature T. Most of the observable Universe is very far from thermal equilibrium, but three thermal bumps can be seen in the cosmic radiation background: optical (stars), infrared (dust) and microwave (CMB). The CMB spectrum measured by COBE is astonishingly close to the Planck function for $T \simeq 2.725K$.

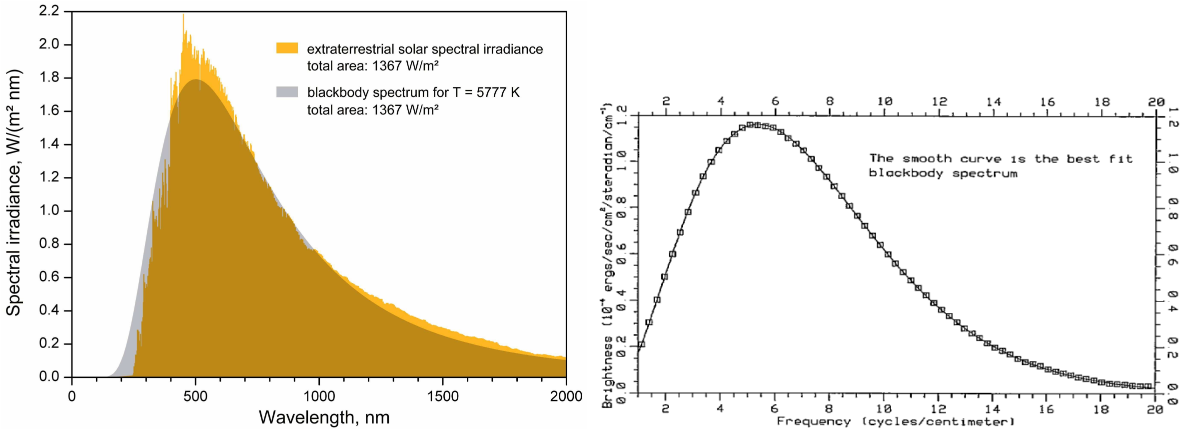

Figure 2: Left: spectrum of the solar radiation (Smerlak 2011, European Journal of Physics, 32, 1143). Right: spectrum of the cosmic microwave background measured by COBE/FIRAS with exposure of 9 minutes (Mather et al. 1990, ApJ, 354, L37).

Planck spectrum. The Planck spectrum can be derived from two principles: (1) the quantization of possible photon states, and (2) statistical distribution of state occupation. Photons are massless bosons that achieve thermal equilibrium with a surrounding system by being constantly emitted and absorbed. Each photon state can be occupied by multiple photons, the distribution of state occupation numbers is determined by thermodynamics.

In a 1-dimensional cavity of length L, the allowed photon states are those of wavelengths $\lambda_m = L/m$ for integers $m = 1, 2, \ldots$. The corresponding energies are $E_m = h\nu_m = hc/\lambda_m = mhc/L$. Each state occupies an element of phase space $\Delta x\,\Delta p_x = \lambda_m E_m/c = h$, in accordance with the uncertainty principle. Each such phase space element allows for 2 states, accounting for the photon spin values $\pm\hbar$. Thus, in a 3-dimensional cavity, the phase space density of states is $\mathrm{d}N_\mathrm{s} = (2/h^3)\mathrm{d}^3x\,\mathrm{d}^3p$. Since $\mathrm{d}^3p = p^2\,\mathrm{d}p\,\mathrm{d}\Omega = (h/c)^3\nu^2\,\mathrm{d}\nu\,\mathrm{d}\Omega$, the spectral density of states is $\mathrm{d}N_\mathrm{s} = (2\nu^2/c^3)\mathrm{d}V\,\mathrm{d}\nu\,\mathrm{d}\Omega$.

Under thermal equilibrium with temperature T, the probability that a particular state is occupied by $n \in \{0, 1, 2, \ldots\}$ photons is $P_n = C\exp(-nE/k_\mathrm{B}T) \equiv CQ^n$ with $Q = \exp(-E/k_\mathrm{B}T)$, where $E = h\nu$ is the energy of a single photon in that state, and k_B is the Boltzmann's constant. The normalization constant C can be found from $\sum_{n=0}^{\infty} P_n = 1$, i.e., $C = 1 - Q$. The *mean occupation number* of a state is:

$$\langle n\rangle = \sum_{n=0}^{\infty} nP_n = \frac{Q}{1-Q} = \frac{1}{\exp(E/k_\mathrm{B}T) - 1}\,. \tag{5.1}$$

The mean spectral number density of photons is thus:

$$\mathrm{d}N = \langle n\rangle\,\mathrm{d}N_\mathrm{s} = \frac{2\nu^2/c^3}{\exp(h\nu/k_\mathrm{B}T) - 1}\,\mathrm{d}V\,\mathrm{d}\nu\,\mathrm{d}\Omega\,. \tag{5.2}$$

This corresponds to a specific intensity:

$$I_\mathrm{th}(\nu, T) = \frac{E\,\mathrm{d}N}{\mathrm{d}V\,\mathrm{d}\nu\,\mathrm{d}\Omega/c} = \frac{2h\nu^3/c^2}{\exp(h\nu/k_\mathrm{B}T) - 1}\,. \tag{5.3}$$

Rayleigh-Jeans law ($h\nu \ll k_\mathrm{B}T$). In the classical limit of low photon energies, one can approximate the denominator to obtain:

$$I_\mathrm{th}(\nu, T) \simeq 2k_\mathrm{B}T\frac{\nu^2}{c^2}\,. \tag{5.4}$$

This can be used to define the *brightness temperature* $T_\mathrm{b} = (I_\nu/2k_\mathrm{B})(c^2/\nu^2)$ as a measure of brightness, especially in radio astronomy.

Wien law ($h\nu \gg k_\mathrm{B}T$). In the limit of high photon energies, the approximation is:

$$I_\mathrm{th}(\nu, T) \simeq \frac{2h\nu^3}{c^2}\exp\left(-\frac{h\nu}{k_\mathrm{B}T}\right)\,. \tag{5.5}$$

Spectral peak. To find the peak of $I_{\rm th}(\nu,T)$, we can write it as function of $x \equiv h\nu/k_{\rm B}T$, i.e., $I_{\rm th}(x,T) = (2/c^2h^3)(k_{\rm B}T)^4x^3/(e^x-1)$, and solve ${\rm d}I_{\rm th}(x,T)/{\rm d}x = 0$, which is equivalent to $(3-x)e^x = 3$, the solution to which is $x_{\rm peak} = h\nu_{\rm peak}/k_{\rm B}T \simeq 2.82$. This relation ($\nu_{\rm peak} \propto T$) is the *Wien displacement law*.

Broad-band astronomical observations are often presented in the form of *spectral energy distribution* (SED) νF_ν, which measures the radiative energy content per decade of frequency range ${\rm d}\mathcal{E}/{\rm d}(\log\nu)$. The peak of a Planck SED is equivalent to solving $(4-x)e^x = 4$, which yields $x_{\rm peak} = h\nu_{\rm peak}/k_{\rm B}T \simeq 3.92$.

Total intensity. The bolometric blackbody intensity can be calculated by integration:

$$I_{\rm th}(T) = \int_0^\infty {\rm d}\nu\, I_{\rm th}(\nu,T) = \frac{2(k_{\rm B}T)^4}{c^2h^3}\int_0^\infty {\rm d}x\,\frac{x^3}{e^x-1}\,. \tag{5.6}$$

The value of this integral is $\pi^4/15$, hence:

$$I_{\rm th}(T) = \frac{2\pi^4k_{\rm B}^4}{15c^2h^3}T^4\,. \tag{5.7}$$

This gives the *Stefan-Boltzmann law*, which is typically stated in terms of emergent energy flux density $F_{\rm th}(T) = \pi I_{\rm th}(T) = \sigma_{\rm SB}T^4$ with the Stefan-Boltzmann constant:

$$\sigma_{\rm SB} = \frac{2\pi^5k_{\rm B}^4}{15c^2h^3}\,. \tag{5.8}$$

The Stefan-Boltzmann law can be used to define the *effective temperature* $T_{\rm eff} = (F/\sigma_{\rm SB})^{1/4}$, using an observational estimate of bolometric energy flux density F.

6 Radiation transport

Recall that the fundamental radiation measure is the specific intensity $I_\nu \equiv {\rm d}\mathcal{E}/({\rm d}A_\perp\,{\rm d}t\,{\rm d}\nu\,{\rm d}\Omega)$, closely related to the phase space density of radiation energy. The total intensity is $I = \int{\rm d}\nu\,I_\nu$. It is of prime astrophysical interest to know how does this measure evolve along the radiation path (geodesic) from the source to the observer.

Emission. Let ${\rm d}l = c\,{\rm d}t$ be a distance element passed by a radiation beam through an emitting medium. An increase of total intensity ${\rm d}I = +j\,{\rm d}l$ defines the *emission coefficient* $j \equiv {\rm d}\mathcal{E}_{\rm em}/({\rm d}t\,{\rm d}V\,{\rm d}\Omega)$, the radiation energy emitted per unit time, unit volume and unit solid angle.

Absorption. A medium through which a radiation beam passes may also absorb some of it, reducing the intensity. A decrease of total intensity ${\rm d}I = -\alpha I\,{\rm d}l$ defines the *absorption coefficient* α [1/cm], a fraction of intensity absorbed per unit length.

A simple model for absorbing medium is that it consists of microscopic absorbers of number density n [1/cm^3], each of a *cross section* σ [cm^2], so that $\alpha = \sigma n$.

Scattering. Scattering is a process of changing the direction of photons. It may be included in both the absorption of an incident beam and emission along scattered beams, or treated separately.

Radiative transfer equation. The combined effect of emission and absorption is described by the *radiative transfer equation*:

$$\frac{{\rm d}I}{{\rm d}l} = j - \alpha I\,. \tag{6.1}$$

Two basic cases are:

- *pure uniform emission*: $I(l) = I(0) + jl$;
- *pure uniform absorption*: $I(l) = I(0)\exp(-\alpha l)$.

Optical depth. For an absorbing medium ($\alpha > 0$), a dimensionless parameter called *optical depth* is defined as $\mathrm{d}\tau = \alpha\,\mathrm{d}l$. It allows to simplify the radiation transfer equation:

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = \frac{j}{\alpha} - I \equiv S - I\,, \tag{6.2}$$

where $S \equiv j/\alpha$ is the *source function*.

Mean free path. The average distance l_{mfp} travelled by a photon before being absorbed (or scattered) corresponds to an optical depth of unity $\tau(l_{\mathrm{mfp}}) = 1$. For a uniform absorption coefficient: $l_{\mathrm{mfp}} = 1/\alpha$.

6.1 Radiative diffusion.

In a relatively opaque homogeneous static medium (high absorption coefficient, short mean free path) of temperature T, any initial radiation field approaches isotropic thermal distribution, i.e., both the specific intensity and source function $I(\nu), S(\nu) \to I_{\mathrm{th}}(\nu, T)$.

Eddington approximation. A weakly anisotropic radiation field is approximated as $I(\mu) \simeq I_0 + I_1\mu$. One can define the following moments of $I(\mu)$:

$$J = \frac{1}{2}\int_{-1}^{1}\mathrm{d}\mu\,I(\mu) = I_0\,, \qquad H = \frac{1}{2}\int_{-1}^{1}\mathrm{d}\mu\,\mu I(\mu) = \frac{I_1}{3}\,, \qquad K = \frac{1}{2}\int_{-1}^{1}\mathrm{d}\mu\,\mu^2 I(\mu) = \frac{I_0}{3}\,. \tag{6.3}$$

These moments are proportional to mean intensity, energy flux density and pressure, respectively. Note that $K = J/3$.

One can take the moments of the radiative transfer equation. Assume that the source function $S = S_0$ is independent of μ (isotropic). Also, define the normal optical depth as $\mathrm{d}\tau_r \equiv \mu\,\mathrm{d}\tau = \mu\alpha\,\mathrm{d}l = \alpha\,\mathrm{d}r$:

$$\mu\frac{\mathrm{d}I}{\mathrm{d}\tau_r} = S - I\,, \tag{6.4}$$

$$\frac{\mathrm{d}H}{\mathrm{d}\tau_r} = \frac{1}{2}\int_{-1}^{1}\mathrm{d}\mu\,\mu\frac{\mathrm{d}I}{\mathrm{d}\tau_r} = \frac{1}{2}\int_{-1}^{1}\mathrm{d}\mu\,(S - I) = S_0 - I_0\,, \tag{6.5}$$

$$\frac{\mathrm{d}K}{\mathrm{d}\tau_r} = \frac{1}{2}\int_{-1}^{1}\mathrm{d}\mu\,\mu^2\frac{\mathrm{d}I}{\mathrm{d}\tau_r} = \frac{1}{2}\int_{-1}^{1}\mathrm{d}\mu\,\mu(S - I) = -H\,, \tag{6.6}$$

$$\frac{1}{3}\frac{\mathrm{d}^2 I_0}{\mathrm{d}\tau_r^2} = \frac{\mathrm{d}^2 K}{\mathrm{d}\tau_r^2} = -\frac{\mathrm{d}H}{\mathrm{d}\tau_r} = I_0 - S_0\,. \tag{6.7}$$

This is a diffusion equation for the isotropic intensity component I_0.

Rosseland approximation. Under a small departure from homogeneity, e.g., a temperature gradient $\mathrm{d}T/\mathrm{d}r$ (e.g. in stellar interiors), radiation transport takes the form of slow *diffusion* by random walks over distances $\sim l_{\mathrm{mfp}}$. The radiation transfer equation can be approximated as:

$$I = S - \frac{\mathrm{d}I}{\mathrm{d}\tau} \simeq I_{\mathrm{th}} - \frac{\mathrm{d}I_{\mathrm{th}}}{\mathrm{d}\tau}\,. \tag{6.8}$$

Consider that $\mathrm{d}\tau = \alpha\,\mathrm{d}l = (\alpha/\mu)\,\mathrm{d}r$, where $\mu = \cos\theta$ of the angle between directions of the radiation beam $\hat{l}$ and of the temperature gradient $\hat{r}$:

$$I \simeq I_{\mathrm{th}} - \frac{\mu}{\alpha}\frac{\mathrm{d}I_{\mathrm{th}}}{\mathrm{d}r}\,. \tag{6.9}$$

Hence, we introduced an anisotropic (dipole) correction which is going to yield a net energy flux density.

$$F(r) = \int_{4\pi}\mathrm{d}\Omega\,I\mu \simeq -\int_{4\pi}\mathrm{d}\Omega\,\frac{\mu^2}{\alpha}\frac{\mathrm{d}I_{\mathrm{th}}}{\mathrm{d}r} = -\frac{2\pi}{\alpha}\frac{\mathrm{d}I_{\mathrm{th}}}{\mathrm{d}r}\int_{-1}^{1}\mathrm{d}\mu\,\mu^2 = -\frac{4\pi}{3\alpha}\frac{\mathrm{d}I_{\mathrm{th}}}{\mathrm{d}r} \tag{6.10}$$

The gradient of thermal intensity can be related to the gradient of temperature using the Stefan-Boltzmann law:

$$I_{\mathrm{th}} = \frac{\sigma_{\mathrm{SB}}T^4}{\pi}\,, \qquad \frac{\mathrm{d}I_{\mathrm{th}}}{\mathrm{d}r} = \frac{\mathrm{d}I_{\mathrm{th}}}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{4\sigma_{\mathrm{SB}}T^3}{\pi}\frac{\mathrm{d}T}{\mathrm{d}r}\,. \tag{6.11}$$

With this, the energy flux density is:

$$F(r) \simeq -\frac{16\sigma_{\rm SB}T^3}{3\alpha}\frac{{\rm d}T}{{\rm d}r}\,. \tag{6.12}$$

This is the *Rosseland approximation for the radiative energy flux density.*

In the above derivation, we have implicitly assumed that absorption coefficient α is independent of radiation frequency. The effect of frequency-dependent α_ν can be introduced at the stage of Eq. (6.10):

$$F(\nu, r) \simeq -\frac{4\pi}{3\alpha_\nu}\frac{{\rm d}I_{\rm th}(\nu, T)}{{\rm d}r} = -\frac{4\pi}{3\alpha_\nu}\frac{\partial I_{\rm th}(\nu, T)}{\partial T}\frac{{\rm d}T}{{\rm d}r}\,. \tag{6.13}$$

Integrating over frequency:

$$F(r) = \int_0^\infty {\rm d}\nu\, F(\nu, r) \simeq -\frac{4\pi}{3}\frac{{\rm d}T}{{\rm d}r}\int_0^\infty {\rm d}\nu\,\alpha_\nu^{-1}\frac{\partial I_{\rm th}(\nu, T)}{\partial T} \equiv -\frac{4\pi}{3}\frac{{\rm d}T}{{\rm d}r}\alpha_{\rm R}^{-1}\frac{{\rm d}I_{\rm th}(T)}{{\rm d}T} = -\frac{16\sigma_{\rm SB}T^3}{3\alpha_{\rm R}}\frac{{\rm d}T}{{\rm d}r}\,, \tag{6.14}$$

where $\alpha_{\rm R}$ is the *Rosseland mean absorption coefficient.*

7 Radiation from moving charges

Source of electromagnetic fields. In lecture 1 we have mentioned that electromagnetic fields in vacuum satisfy wave equations. In general, the Maxwell's equations couple the electromagnetic fields to electric charges q_i of volume density $\rho_{\rm e} = \sum_i q_i n_i$ and currents $q_i\vec{v}_i$ of density $\vec{j} = \sum_i q_i\vec{v}_i$:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho_{\rm e}\,, \qquad c\vec{\nabla}\times\vec{B} - \partial_t\vec{E} = 4\pi\vec{j}\,. \tag{7.1}$$

Electromagnetic potentials. A symmetry of these equations can be revealed by using the *electromagnetic potentials.* The *electric scalar potential* ϕ and magnetic vector potential $\vec{A}$ can always be found to satisfy $\vec{E} = -\vec{\nabla}\phi - \partial_0\vec{A}$ [4] and $\vec{B} = \vec{\nabla}\times\vec{A}$. Such potentials automatically satisfy the remaining two Maxwell's equations $\vec{\nabla}\cdot\vec{B} = 0$ and $\vec{\nabla}\times\vec{E} = -\partial_0\vec{B}$. They are, however, not unique: potentials modified as $\vec{A} + \vec{\nabla}\psi$ and $\phi - \partial_0\psi$ for any *gauge function* produce the same electromagnetic fields.

Electromagnetic potential wave equations. In terms of the potentials, Eqs. (7.1) become:

$$\partial_0^2\phi - \nabla^2\phi = 4\pi\rho_{\rm e}\,, \qquad \partial_0^2\vec{A} - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{j}\,, \tag{7.2}$$

together with the *Lorentz gauge* $\partial_0\phi + \vec{\nabla}\cdot\vec{A} = 0$.

Note an elegant covariance of these equations in terms of the 4-vectors $A^\mu = (\phi, A^i)$, $j^\mu = (\rho_{\rm e}c, j^i)$, and $\partial_\mu = (\partial_0, \partial_i)$:

$$\partial^\nu\partial_\nu A^\mu = \frac{4\pi}{c}j^\mu\,, \qquad \partial_\mu A^\mu = 0\,. \tag{7.3}$$

Retarded potentials. Eqs. (7.2) can be solved using a Green's function to obtain:

$$A^\mu(t, \vec{r}) = \frac{1}{c}\int {\rm d}t'\,{\rm d}^3\vec{r}\,'\,\frac{j^\mu(t', \vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\,\delta\left(t' - t + |\vec{r} - \vec{r}\,'|/c\right)\,, \tag{7.4}$$

where the δ function selects the *retarded time* $t' = t - |\vec{r} - \vec{r}\,'|/c$ for any *retarded position* $\vec{r}\,'$.

This approach is particularly meaningful in the astronomical (especially cosmological) context - the electromagnetic radiation that we presently observe was produced by the distribution of electric charge in often extremely distant sources in their distant past.

[4] $\partial_0 \equiv \partial/\partial x^0 = \partial/\partial(ct) = (1/c)\partial_t$

Retarded potentials from a single charged particle. As the most elementary example, one can consider a single particle of charge q, trajectory $\vec{r}_0(t)$, and velocity $\vec{v}_0(t) = \mathrm{d}\vec{r}_0/\mathrm{d}t$. The contribution of this particle to the charge and current densities can be expressed in terms of the δ function:

$$\rho_\mathrm{e}(t,\vec{r}) = q\,\delta\left(\vec{r} - \vec{r}_0(t)\right)\,, \qquad \vec{j}(t,\vec{r}) = q\vec{v}_0(t)\,\delta\left(\vec{r} - \vec{r}_0(t)\right)\,. \tag{7.5}$$

Let's calculate the electric scalar potential:

$$\phi(t,\vec{r}) = q\int \mathrm{d}t'\,\mathrm{d}^3\vec{r}'\,\frac{\delta\left(\vec{r}' - \vec{r}_0(t')\right)}{|\vec{r} - \vec{r}'|}\,\delta\left(t' - t + |\vec{r} - \vec{r}'|/c\right)\,. \tag{7.6}$$

The first δ function can be eliminated by integrating over $\mathrm{d}^3\vec{r}'$:

$$\phi(t,\vec{r}) = q\int \mathrm{d}t'\,\frac{\delta(t' - t + R(t',\vec{r})/c)}{R(t',\vec{r})}\,, \tag{7.7}$$

where $\vec{R}(t',\vec{r}) = \vec{r} - \vec{r}_0(t')$ and $R = |\vec{R}|$.

The argument of the remaining δ function is a non-linear function of t'. It is substituted as $t'' = t' - t + R(t',\vec{r})/c$, hence $\mathrm{d}t'' = [1 - \vec{n}(t',\vec{r})\cdot\vec{\beta}_0(t')]\,\mathrm{d}t' \equiv \kappa(t',\vec{r})\,\mathrm{d}t'$, where $\vec{n}(t',\vec{r}) = \vec{R}/|\vec{R}|$ and $\vec{\beta}_0 = \vec{v}_0/c$:

$$\phi(t,\vec{r}) = q\int \mathrm{d}t''\,\frac{\delta(t'')}{\kappa(t',\vec{r})\,R(t',\vec{r})} = \frac{q}{\kappa(t',\vec{r})\,R(t',\vec{r})}\,. \tag{7.8}$$

If the particle is at rest ($\vec{\beta}_0 = 0$, hence $\kappa = 1$), this result is consistent with the Coulomb potential. The additional factor $\kappa(t',\vec{r})$ contributes to the relativistic beaming (Doppler effect and aberration).

The corresponding magnetic vector potential is

$$\vec{A}(t,\vec{r}) = \frac{q\vec{\beta}_0(t')}{\kappa(t',\vec{r})\,R(t',\vec{r})}\,. \tag{7.9}$$

Together, these are known as the *Lienard-Wiechert potentials.*

Retarded electromagnetic fields. The Lienard-Wiechert potentials correspond to the following electric field:

$$\vec{E}(t,\vec{r}) = \frac{q}{\Gamma^2\kappa^3R^2}\left(\vec{n} - \vec{\beta}\right) + \frac{q}{c^2\kappa^3R}\left\{\vec{n}\times\left[\left(\vec{n} - \vec{\beta}\right)\times\vec{a}\right]\right\}\,, \tag{7.10}$$

where $\Gamma = 1/\sqrt{1-\beta^2}$ is the Lorentz factor, and $\vec{a} = \mathrm{d}\vec{v}/\mathrm{d}t$ is the acceleration. All of the right-hand side should be evaluated at the retarded time t'. The corresponding magnetic field is simply $\vec{B}(t,\vec{r}) = \vec{n}(t',\vec{r})\times\vec{E}(t,\vec{r})$.

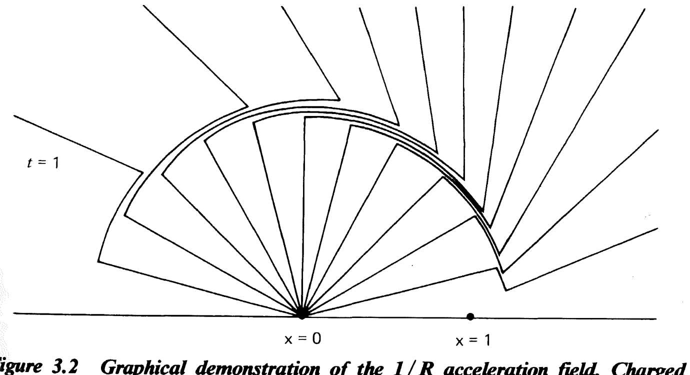

Figure 3.2 Graphical demonstration of the 1/R acceleration field. Charged particle moving at uniform velocity in positive x direction is stopped at x = 0 and t = 0.

Figure 3: (Fig 3.2 in Rybicki & Lightman, 1979) Electric field lines sourced by a charge shifting rapidly from resting at $x = 1$ to resting at $x = 0$. The $1/R^2$ Coulomb fields must be joined within a transition shell of fixed width, hence the field strength within the shell scales like $1/R$. This argument was originally presented by Thomson (1906).

This electromagnetic field consists of two components. The first term (*velocity field*) scales like $1/R^2$ and reduces to the Coulomb field in the limit of charge at rest. The second term (*acceleration* *field*) scales like $1/R$, clearly dominating at large distances, and it is strictly perpendicular to $\vec{n}$. This term is responsible for the electromagnetic radiation:

$$\vec{E}_{\rm rad}(t,\vec{r}) = \frac{q}{c^2\kappa^3 R}\left\{\vec{n}\times\left[\left(\vec{n}-\vec{\beta}\right)\times\vec{a}\right]\right\}. \tag{7.11}$$

Non-relativistic limit. When the particle velocity is non-relativistic ($\beta \ll 1$), beaming becomes unimportant ($\kappa \simeq 1$). The radiation field is simplified to:

$$\vec{E}_{\rm rad}(t,\vec{r}) \simeq \frac{q}{c^2 R}\left\{\vec{n}\times[\vec{n}\times\vec{a}]\right\}. \tag{7.12}$$

This vector product can be recognized to represent the component of acceleration vector perpendicular to the line of sight $\vec{n}$. Let Θ be the angle between $\vec{a}$ and $\vec{n}$. Since the parallel acceleration is $\vec{a}_\parallel = (\vec{a}\cdot\vec{n})\vec{n} = (a\cos\Theta)\vec{n}$, the perpendicular acceleration is $\vec{a}_\perp = \vec{a}-\vec{a}_\parallel = (\vec{n}\cdot\vec{n})\vec{a}-(\vec{a}\cdot\vec{n})\vec{n} = -\vec{n}\times(\vec{n}\times\vec{a})$, its magnitude is $a_\perp = a\sin\Theta$, hence:

$$\vec{E}_{\rm rad}(t,\vec{r}) \simeq -\frac{q}{c^2 R}\vec{a}_\perp\,, \qquad |E_{\rm rad}| \simeq \frac{|qa_\perp|}{c^2 R}. \tag{7.13}$$

This electric field is a *dipole*, and its Poynting flux density is then:

$$S = \frac{c}{4\pi}E_{\rm rad}^2 \simeq \frac{1}{4\pi R^2}\frac{q^2}{c^3}a_\perp^2\,. \tag{7.14}$$

This represents the energy flux density $\mathrm{d}\mathcal{E}/(\mathrm{d}A_\perp\,\mathrm{d}t)$ of a unidirectional radiation beam through a normal area element $\mathrm{d}A_\perp = R^2\,\mathrm{d}\Omega$. The total luminosity emitted by a non-relativistic charge is thus:

$$L = \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \int \mathrm{d}A_\perp\,S \simeq \int \mathrm{d}\Omega\,\frac{q^2}{4\pi c^3}a_\perp^2 = \frac{q^2 a^2}{4\pi c^3}\int \mathrm{d}\Omega\,\sin^2\Theta = \frac{q^2a^2}{2c^3}\int_{-1}^{1}\mathrm{d}\mu\,(1-\mu^2) = \frac{2q^2}{3c^3}a^2\,. \tag{7.15}$$

This is known as the *Larmor's formula*. Radiation from a non-relativistic charge is a dipole perpendicular to the acceleration vector.

Spectrum. A Fourier transform of the radiation electric field is:

$$\hat{E}_{\rm rad}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}t\,\exp(i\omega t)\,E_{\rm rad}(t)\,. \tag{7.16}$$

If, however, the radiation field has a form of a sharp pulse of duration τ, the transform can be limited to the frequency window of $\omega\tau < 1$, in which $\exp(i\omega t) \simeq 1$. Then we have:

$$\hat{E}_{\rm rad}(\omega) \simeq \frac{q}{2\pi c^2 R}\int_\tau \mathrm{d}t\,a_\perp(t) = \frac{q}{2\pi c^2 R}\Delta v_\perp\,. \tag{7.17}$$

The energy spectrum per unit area will be:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\omega\,\mathrm{d}A_\perp} = c|\hat{E}(\omega)|^2 \simeq \frac{q^2}{(2\pi)^2c^3R^2}(\Delta v_\perp)^2\,. \tag{7.18}$$

Integrating over $\mathrm{d}A_\perp = R^2\,\mathrm{d}\Omega$ in the dipole approximation $\Delta v_\perp = \Delta v\sin\Theta$, we find the total energy spectrum:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\omega} \simeq \frac{q^2}{(2\pi)^2c^3}(\Delta v)^2\int \mathrm{d}\Omega\,\sin^2\Theta = \frac{2q^2}{3\pi c^3}(\Delta v)^2\,. \tag{7.19}$$

The spectrum of radiation produced by many impulsive acceleration events is thus determined by the statistics of velocity changes during such events.

Dipole approximation. In the case of many charges located in a region much smaller than its distance R_0, i.e., at positions $\vec{R}_i = \vec{R}_0 + \vec{r}_i$ such that $r_i \ll R_0$, the relevant parameter for their collective emission is the dipole moment $\vec{d} = \sum_i q_i\vec{r}_i$ and its second time derivative $\ddot{\vec{d}} = \sum_i q_i\vec{a}_i$.

Relativistic limit. In special relativity, the four-acceleration can be defined as $a^\mu = \mathrm{d}u^\mu/\mathrm{d}\tau = \mathrm{d}^2x^\mu/\mathrm{d}\tau^2$, where τ is the *proper time*. Recall that $x^\mu = (ct, \vec{x})$, $u^\mu = \gamma(c, \vec{v})$, $\gamma = (1 - v^2/c^2)^{-1/2}$. In particular, $u^0 = \mathrm{d}(ct)/\mathrm{d}\tau = \gamma c$, hence $\mathrm{d}\tau = \mathrm{d}t/\gamma$. Since $u_\mu u^\mu = -c^2$, then $a_\mu u^\mu = (1/2)\mathrm{d}(u_\mu u^\mu)/\mathrm{d}\tau = 0$. In the particle's instantaneous rest frame, $u'^\mu = (c, 0, 0, 0)$, hence $a'_\mu u'^\mu = a'_0 c = 0$, hence $a'_0 = 0$, hence $a_\mu a^\mu = a'_\mu a'^\mu = \vec{a}' \cdot \vec{a}'$. Since energy and time transform in the same way, total power or luminosity is a Lorentz invariant: $L = \mathrm{d}\mathcal{E}/\mathrm{d}t = \mathrm{d}\mathcal{E}'/\mathrm{d}t' = L'$.

The Larmor's formula can thus be generalized to a covariant form:

$$L = L' = \frac{2q^2}{3c^3}\,(\vec{a}' \cdot \vec{a}') = \frac{2q^2}{3c^3} a_\mu a^\mu\,. \tag{7.20}$$

Lorentz transformation of acceleration is different for the components parallel and perpendicular to the particle velocity:

$$a'_\parallel = \gamma^3 a_\parallel\,, \qquad a'_\perp = \gamma^2 a_\perp\,. \tag{7.21}$$

$$L = \frac{2q^2}{3c^3}\left({a'_\parallel}^2 + {a'_\perp}^2\right) = \frac{2q^2}{3c^3}\gamma^4\left(\gamma^2 a_\parallel^2 + a_\perp^2\right)\,. \tag{7.22}$$

8 Bremsstrahlung

This German term literally means *deceleration radiation*, it is the radiation of unbound charge decelerating in the Coulomb electric potential of another charge, also known as the *free-free emission*. The most relevant astrophysical case is that of an electron ($q = -e$) interacting with an ion ($q = Ze$). Because of the high mass ratio, the ion can be considered fixed.

Single interaction. A classical treatment of the problem is valid in the limit of small scattering angle. Let $v_0 \ll c$ be the initial velocity of the electron approaching the ion with impact parameter b. Those two parameters define the *collision timescale* $\tau = b/v_0$, and two energy scales: kinetic $m_\mathrm{e}v_0^2/2$ and electrostatic potential Ze^2/b.

Electron velocity change (I). As an order-of-magnitude appoximation, consider the peak Lorentz force $F_\mathrm{peak} = Ze^2/b^2$ acting on time τ changes the electron velocity by roughly

$$\Delta v = \frac{\Delta p}{m_\mathrm{e}} \sim \frac{(Ze^2/b^2)\tau}{m_\mathrm{e}} = \frac{Ze^2}{m_\mathrm{e} b v_0}\,. \tag{8.1}$$

Electron velocity change (II). Noticing a planar symmetry of the problem, one can consider a coordinate system (x, y) centered on the ion, in which the electron trajectory is $\vec{R}_\mathrm{e}(t) = (v_0 t, b) = b(t/\tau, 1)$. The electron distance follows a universal time profile $R_\mathrm{e}(t)/b = (t^2/\tau^2 + 1)^{1/2}$. This corresponds to a universal time profile of acceleration (parallel or perpendicular to $\vec{v}_0$). In the case of perpendicular acceleration $a_\perp = a_\mathrm{peak}(R_\mathrm{e}/b)^{-3}$, where $a_\mathrm{peak} = F_\mathrm{peak}/m_\mathrm{e} = Ze^2/(m_\mathrm{e}b^2)$, it can be shown that the velocity change amounts to

$$\Delta v = 2a_\mathrm{peak}\tau = \frac{2Ze^2}{m_\mathrm{e} b v_0}\,. \tag{8.2}$$

Total luminosity. The peak acceleration $a_\mathrm{peak} = Ze^2/(m_\mathrm{e}b^2)$ can be put into the Larmor's formula to find the peak total luminosity that a single particle can produce:

$$L_\mathrm{peak} = \frac{2e^2}{3c^3}\left(\frac{Ze^2}{m_\mathrm{e}b^2}\right)^2 = \frac{2Z^2e^6}{3m_\mathrm{e}^2c^3b^4} = \frac{2}{3}cr_\mathrm{e}^2\left(\frac{Ze}{b^2}\right)^2\,, \tag{8.3}$$

where $r_\mathrm{e} = e^2/(m_\mathrm{e}c^2)$ is the classical electron radius.

Spectrum (I). The core of the acceleration profile is close to a Gaussian with dispersion $\Delta t = \tau$. Its Fourier transform is thus close to a Gaussian with dispersion $\Delta\omega = 2/\tau$. The total emitted energy $\Delta\mathcal{E} \sim L_\mathrm{peak}\tau$ is now allocated into a spectral window $\nu \lesssim \Delta\nu = 1/(\pi\tau)$: $\Delta\mathcal{E} \sim \mathcal{E}(\nu)\,\Delta\nu$, hence:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\nu} \sim \tau^2 L_\mathrm{peak} \sim \frac{Z^2e^6}{m_\mathrm{e}^2c^3b^2v_0^2} = cr_\mathrm{e}^2\left(\frac{Ze}{bv_0}\right)^2\,. \tag{8.4}$$

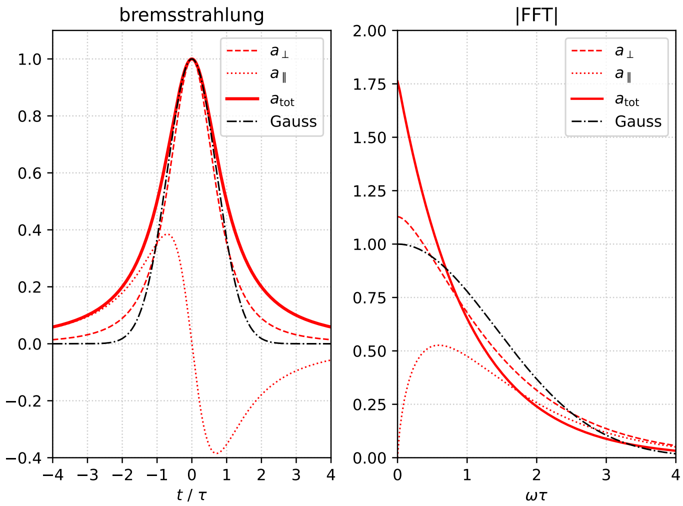

Figure 4: Left panel: time profiles of electron acceleration (distinguishing components parallel or perpendicular to the electron velocity direction) in the Coulomb electric field of an ion at rest (bremsstrahlung). Rigth panel: Fourier transforms of the acceleration profiles into the frequency space.

Spectrum (II). Using the formula for energy spectrum in the dipole approximation:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\nu} \simeq \frac{4q^2}{3c^3}(\Delta v)^2 = \frac{16}{3}\frac{Z^2e^6}{m_\mathrm{e}^2c^3b^2v_0^2} = \frac{16}{3}\tau^2 L_\mathrm{peak}\,. \tag{8.5}$$

Example. The velocity change is significant, $\Delta v \sim v_0$, when electrostatic potential energy is comparable to the kinetic energy. For thermal ionized hydrogen ($Z = 1$) with $m_\mathrm{e}v_0^2 \sim k_\mathrm{B}T$ and temperature $T = 10^4 T_4\,\mathrm{K}$, this corresponds to an impact parameter $b \sim Ze^2/(k_\mathrm{B}T) \sim 2ZT_4^{-1}\,\mathrm{nm}$, collision timescale $\tau = b/v_0 \sim 10^{-13}T_4^{-3/2}\,\mathrm{s}$, peak luminosity $L_{\mathrm{peak},0} = Z^2e^6/(m_\mathrm{e}^2c^3b^4) \sim 10^{-6}Z^{-2}T_4^4\,\mathrm{erg/s}$, and characteristic frequency $\Delta\nu \sim 10^{14}Z^{-1}T_4^{3/2}\,\mathrm{Hz}$ (infrared).

Emissivity. Consider a medium consisting of ions of density n_i and electrons of density n_e and velocity v_0. Interactions of electrons with ions happen over a broad range of impact parameter b. The cross section for interactions at $b < b' < b + \mathrm{d}b$ is $2\pi b\,\mathrm{d}b$. Over a time element $\mathrm{d}t$ a single electron interacts with $\mathrm{d}N_\mathrm{i} = n_\mathrm{i}(v_0\mathrm{d}t)(2\pi b\,\mathrm{d}b)$ ions, emitting radiation of spectrum $\mathcal{E}(\nu) = (16/3)Z^2e^6/(m_\mathrm{e}^2c^3b^2v_0^2)$. In a volume element $\mathrm{d}V$ there are $\mathrm{d}N_\mathrm{e} = n_\mathrm{e}\,\mathrm{d}V$ electrons. Hence, the collective radiation of those electrons can be calculated as:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\nu\,\mathrm{d}V\,\mathrm{d}t} = \frac{32\pi}{3}\frac{e^6}{m_\mathrm{e}^2c^3}\frac{Z^2n_\mathrm{e}n_\mathrm{i}}{v_0}\int_{b_\mathrm{min}}^{b_\mathrm{max}}\frac{\mathrm{d}b}{b}\,. \tag{8.6}$$

At a given frequency ν, only interactions with impact parameter $b \lesssim b_\mathrm{max} = v_0/(\pi\nu)$ contribute. The minimum value of impact parameter is effectively $b_\mathrm{min} = h/(m_\mathrm{e}v_0)$.

An accurate result is parametrized with a *Gaunt factor* $g_\mathrm{ff}(v_0,\nu)$ such that:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\nu\,\mathrm{d}V\,\mathrm{d}t} = \frac{32\pi^2}{3\sqrt{3}}\frac{e^6}{m_\mathrm{e}^2c^3}\frac{Z^2n_\mathrm{e}n_\mathrm{i}}{v_0}g_\mathrm{ff}\,. \tag{8.7}$$

9 Radiation from electrons accelerated in magnetic field

9.1 Cyclotron radiation

Electron acceleration in uniform magnetic field. In the presence of uniform magnetic field, e.g. $\vec{B} = B_0\hat{z}$, the Lorentz force on an electron with $q = -e$ and velocity $\vec{v} = \vec{\beta}c$ is $\vec{F}_\mathrm{L} = q\vec{\beta}\times\vec{B}$. Acceleration thus affects only the perpendicular velocity component $\vec{v}_\perp = v\sin\alpha$, where α is the *pitch angle*. Neglecting any motion along the field line (it yields no acceleration), in the plane perpendicular to $\vec{B}$, here (x, y), the electron moves along a circle with the *gyroradius* (or Larmor radius) $R_{\rm L}$, for example:

$$\vec{r}(t) = R_{\rm L}\left[\hat{x}\sin(\Omega_{\rm L}t) - \hat{y}\cos(\Omega_{\rm L}t)\right] \equiv R_{\rm L}\hat{r}\,, \tag{9.1}$$

$$\vec{v}_\perp(t) = \frac{{\rm d}\vec{r}}{{\rm d}t} = R_{\rm L}\Omega_{\rm L}\left[\hat{x}\cos(\Omega_{\rm L}t) + \hat{y}\sin(\Omega_{\rm L}t)\right] \equiv v_\perp\hat{v} \equiv \beta_\perp c\hat{v}\,, \tag{9.2}$$

$$\vec{a}(t) = \frac{{\rm d}\vec{v}}{{\rm d}t} = -R_{\rm L}\Omega_{\rm L}^2\left[\hat{x}\sin(\Omega_{\rm L}t) - \hat{y}\cos(\Omega_{\rm L}t)\right] \equiv -a_{\rm L}\hat{r}\,, \tag{9.3}$$

where $\Omega_{\rm L}$ is the *gyrofrequency* (or Larmor frequency), which can be calculated from the ratio of acceleration and velocity amplitudes:

$$\Omega_{\rm L} = \frac{a_{\rm L}}{v_\perp} = \frac{F_{\rm L}}{m_{\rm e}v_\perp} = \frac{e\beta_\perp B_0}{m_{\rm e}v_\perp} = \frac{eB_0}{m_{\rm e}c}\,. \tag{9.4}$$

The gyroradius can be then calculated from the velocity:

$$R_{\rm L} = \frac{v_\perp}{\Omega_{\rm L}} = \frac{m_{\rm e}cv_\perp}{eB_0}\,. \tag{9.5}$$

Cyclotron radiation. Acceleration of a non-relativistic particle in uniform magnetic field produces the *cyclotron radiation.*

Consider an observer located at large distance R in the direction $\hat{n} = \hat{z}\cos\Theta + \hat{x}\sin\Theta$. In order to determine the radiation electric field $\vec{E}_{\rm rad} \simeq e\vec{a}_\perp/(c^2R)$, we need to find the perpendicular acceleration vector $\vec{a}_\perp \perp \hat{n}$. The parallel acceleration is $a_\parallel = \vec{a}\cdot\hat{n} = -a_{\rm L}\sin(\Omega_{\rm L}t)\sin\Theta$. The perpendicular acceleration is:

$$\vec{a}_\perp = \vec{a} - a_\parallel\hat{n} = a_{\rm L}\left[-\sin(\Omega_{\rm L}t)\cos^2\Theta, \cos(\Omega_{\rm L}t), \sin(\Omega_{\rm L}t)\sin\Theta\cos\Theta\right]\,. \tag{9.6}$$

Angular distribution of emitted power. One can show that $a_\perp^2 = a_{\rm L}^2\left[1 - \sin^2(\Omega_{\rm L}t)\sin^2\Theta\right]$. Averaging over time, we have $\left\langle a_\perp^2\right\rangle_t = a_{\rm L}^2\left(1 - \sin^2\Theta/2\right)$. The angular distribution of radiation power is a combination of isotropic and dipole components.

Recall that the Poynting flux density corresponding to the radiation electric field is:

$$S = \frac{c}{4\pi}E_{\rm rad}^2 \simeq \frac{1}{4\pi R^2}\frac{e^2}{c^3}a_\perp^2\,. \tag{9.7}$$

Taking the time-averaged acceleration:

$$\langle S\rangle_t \simeq \frac{1}{4\pi R^2}\frac{e^2}{c^3}a_{\rm L}^2\left(1 - \frac{1}{2}\sin^2\Theta\right)\,. \tag{9.8}$$

Total luminosity. The Larmor acceleration can be expressed in terms of classical electron radius $r_{\rm e} = e^2/(m_{\rm e}c^2)$, Thomson cross section $\sigma_{\rm T} = (8\pi/3)r_{\rm e}^2$, and background magnetic energy density $u_{\rm B0} = B_0^2/8\pi$:

$$a_{\rm L}^2 = \Omega_{\rm L}^2v_\perp^2 = \frac{e^2B_0^2}{m_{\rm e}^2c^2}v_\perp^2 = \frac{c^2}{e^2}r_{\rm e}^2B_0^2v_\perp^2 = \frac{3c^2}{e^2}\frac{8\pi}{3}r_{\rm e}^2\frac{B_0^2}{8\pi}v_\perp^2 = \frac{3c^2}{e^2}\sigma_{\rm T}u_{\rm B0}v_\perp^2\,, \tag{9.9}$$

$$\langle S\rangle_t \simeq \frac{3}{4\pi R^2}c\sigma_{\rm T}u_{\rm B0}\beta_\perp^2\left(1 - \frac{1}{2}\sin^2\Theta\right)\,. \tag{9.10}$$

Recognize that $S = {\rm d}E/({\rm d}t\,{\rm d}A_\perp) = {\rm d}L/(R^2\,{\rm d}\Omega)$, the total luminosity is

$$L = \frac{3}{4\pi}c\sigma_{\rm T}u_{\rm B0}\beta_\perp^2\int_{4\pi}{\rm d}\Omega\left(1 - \frac{1}{2}\sin^2\Theta\right)\,. \tag{9.11}$$

Substituting $\mu = \cos\Theta$ and ${\rm d}\Omega = 2\pi\,{\rm d}\mu$, one can calculate the integral to be $8\pi/3$, hence:

$$L = 2c\sigma_{\rm T}u_{\rm B0}\beta_\perp^2\,. \tag{9.12}$$

For isotropic distribution of electrons, averaging $\beta_\perp^2 = \beta^2\sin^2\theta$ over the pitch angle α gives $\left\langle\beta_\perp^2\right\rangle_\alpha = (2/3)\beta^2$, hence:

$$L_{\rm iso} = \frac{4}{3}c\sigma_{\rm T}u_{\rm B0}\beta^2\,. \tag{9.13}$$

Polarization. Since acceleration is strictly harmonic in time (the frequency spectrum is discrete), radiation for particles with the same Ω_L is completely polarized.

- For an observer located along the magnetic field ($\Theta = 0$), $\vec{a}_\perp$ rotates in the (x, y) plane, the polarization is strictly circular.
- For an observer located perpendicular to the magnetic field ($\Theta = 90°$), $\vec{a}_\perp$ has only the y component, the polarization is strictly linear.

9.2 Synchrotron radiation

When the charged particle (here electron) propagating in uniform magnetic field $\vec{B}_0$ becomes relativistic with velocity $\beta = v/c \lesssim 1$ and Lorentz factor $\gamma = (1-\beta^2)^{-1/2} \gg 1$, the Lorentz force has the same form as in the non-relativistic case, but it causes a change of relativistic momentum $p_\perp = \gamma m_e v_\perp$ perpendicular to $\vec{B}_0$. The gyrofrequency is changed to $\Omega_L = a_\perp/v_\perp = eB_0/(\gamma m_e c)$, and the gyroradius to $R_L = v_\perp/\Omega_L = \gamma m_e c v_\perp/(eB_0)$.

Total emitted luminosity. In the relativistic case it is necessary to distinguish the radiative power emitted into a unit solid angle from the radiative power received from the same solid angle. This is because the emitting particle approaches the observer, chasing the photons emitted previously (light-travel effect). Here we are concerned with the radiative power emitted in all directions, i.e., the total emitted luminosity.

The Larmor's formula can be used in the instantaneous frame of the electron $\mathcal{O}'$, using the invariance of power.

$$L_{\rm em} = L'_{\rm em} = \frac{2e^2}{3c^3}(a')^2 \tag{9.14}$$

Noting that the acceleration vector is strictly perpendicular to the velocity $\vec{a} = \vec{a}_\perp \perp \vec{v}$ (in $\mathcal{O}'$, $\vec{a}'$ is perpendicular to the velocity of $\mathcal{O}$), the Lorentz transformation of acceleration is $a'_\perp = \gamma^2 a_\perp$. One can write that:

$$(a')^2 = \gamma^4 a_\perp^2 = \gamma^4 \Omega_L^2 v_\perp^2 = \gamma^2 \frac{e^2 B_0^2}{m_e^2 c^2} v_\perp^2 \tag{9.15}$$

Compared with the non-relativistic case, the luminosity will be multiplied by γ^2 factor:

$$L_{\rm syn}(\alpha) = 2c\sigma_T u_{B0}\gamma^2\beta^2 \sin^2\alpha\,, \qquad L_{\rm syn,iso} = \frac{4}{3} c\sigma_T u_{B0}\gamma^2\beta^2\,. \tag{9.16}$$

Cooling time scale. The cooling time scale $\tau_{\rm cool}$ is the ratio of electron energy to the emitted luminosity:

$$\tau_{\rm cool,syn} = \frac{\gamma m_e c^2}{L_{\rm syn}(\alpha)} = \frac{1}{4\gamma\beta^2\sin^2\alpha} \frac{m_e c^2}{(4\pi/3) r_e^3} \frac{1}{u_{B0}} \frac{r_e}{c} \equiv \frac{1}{4\gamma\beta^2\sin^2\alpha} \frac{u_e}{u_{B0}} \tau_e\,, \tag{9.17}$$

where $\tau_e \equiv r_e/c \simeq 0.94 \times 10^{-23}$ s is the classical electron light crossing time scale, and $u_e = m_e c^2/[(4\pi/3) r_e^3] \simeq 0.87 \times 10^{31}\,{\rm erg/cm^3}$ is the classical electron energy density. For $\alpha = 90°$ and $\beta \simeq 1$, one has $\tau_{\rm cool,syn} \simeq 0.65\gamma^{-1}\,(u_{B0}[{\rm erg/cm^3}])^{-1}$ yr.

Relativistic beaming. For relativistic particle motion, the expression for radiation electric field $\vec{E}_{\rm rad}$ includes the κ^{-3} term, where $\kappa = 1 - \hat{n}\cdot\vec{\beta}$ represents time retardation. Introducing the emission angle $\theta_{\rm em}$ between the emission direction (line of sight) $\hat{n}$ and particle velocity $\vec{\beta} = \vec{v}/c$, one finds $\kappa = 1 - \beta\cos\theta_{\rm em}$. This $\kappa \ll 1$ only when both $\beta \simeq 1$ and $\cos\theta_{\rm em} \simeq 1$. In the relativistic limit, $\beta \simeq 1 - 1/(2\gamma^2)$. In the limit of small angles ($\theta_{\rm em} \ll 1$), $\cos\theta_{\rm em} \simeq 1 - \theta_{\rm em}^2/2$. Taken together, they yield $\kappa \simeq (1 + \gamma^2\theta_{\rm em}^2)/(2\gamma^2)$. Radiation is thus strongly beamed into a cone $\theta_{\rm em} \lesssim 1/\gamma$ around the instantaneous electron velocity.

Characteristic time scale. A relativistic electron that gyrates in the magnetic field with frequency $\Omega_{\rm L}$ emits synchrotron radiation into a very narrow cone sweeping its sky. An observer that is swept by such beamed radiation would detect a very narrow pulse. Consider the case of perpendicular pitch angle $\alpha = 90^\circ$ and observer located at $\hat{n} = \hat{x}$. Let the particle trajectory and velocity be as before:

$$\vec{r}(t) = R_{\rm L}[\hat{x}\sin(\Omega_{\rm L}t) - \hat{y}\cos(\Omega_{\rm L}t)]\,, \tag{9.18}$$

$$\vec{v}(t) = v[\hat{x}\cos(\Omega_{\rm L}t) + \hat{y}\sin(\Omega_{\rm L}t)]\,. \tag{9.19}$$

The emission angle is $\theta_{\rm em}(t) = \Omega_{\rm L}t$. Introduce two emission moments with $\theta_{\rm em}(t_{\rm em,1}) = -1/\gamma$ and $\theta_{\rm em}(t_{\rm em,2}) = 1/\gamma$, hence $t_{\rm em,1} = -1/(\gamma\Omega_{\rm L})$ and $t_{\rm em,2} = 1/(\gamma\Omega_{\rm L})$. The time it takes to emit a single pulse along $\hat{n}$ is $\Delta t_{\rm em} = t_{\rm em,2} - t_{\rm em,1} = 2/(\gamma\Omega_{\rm L})$. However, at $t_{\rm em,1}$ the electron was at $x_{\rm em,1} = R_{\rm L}\sin(-1/\gamma) \simeq -R_{\rm L}/\gamma$, and at $t_{\rm em,2}$ the electron was at $x_{\rm em,2} \simeq R_{\rm L}/\gamma$, the difference being $\Delta x_{\rm em} \simeq 2R_{\rm L}/\gamma = 2v/(\gamma\Omega_{\rm L})$. The observed time scale is shortened by the light travel effect:

$$\Delta t_{\rm obs} = \Delta t_{\rm em} - \frac{\Delta x_{\rm em}}{c} = \frac{2(1-\beta)}{\gamma\Omega_{\rm L}} \simeq \frac{1}{\gamma^3\Omega_{\rm L}}\,. \tag{9.20}$$

Characteristic frequency. The extreme shortness of observed pulses means that the spectrum can extend into very high frequencies. Detailed synchrotron spectrum for arbitrary pitch angle α has a characteristic frequency of $\omega_{\rm c} = (3/2)\gamma^3\Omega_{\rm L}\sin\alpha$.

Spectrum and polarization. Because of the relativistic beaming, synchrotron radiation is not harmonic like cyclotron, but consists of characteristic pulses that result in a continuous spectrum. Detailed calculation of the spectrum distinguishes two polarizations for the radiation electric field: parallel or perpendicular to the magnetic field projected onto the plane normal to the line of sight $\hat{n}$:

$$\frac{{\rm d}L_\perp}{{\rm d}\omega} = \frac{\sqrt{3}e^3B\sin\alpha}{4\pi m_{\rm e}c^2}\,[F(x)+G(x)]\,, \tag{9.21}$$

$$\frac{{\rm d}L_\parallel}{{\rm d}\omega} = \frac{\sqrt{3}e^3B\sin\alpha}{4\pi m_{\rm e}c^2}\,[F(x)-G(x)]\,, \tag{9.22}$$

with special kernel functions of $x = \omega/\omega_{\rm c}$:

$$F(x) = x\int_x^\infty {\rm d}\xi\, K_{5/3}(\xi)\,, \qquad G(x) = x\,K_{2/3}(x)\,. \tag{9.23}$$

The function $F(x)$ represents the spectral shape of the total emission, and the function $G(x)$ represents the spectral shape of the polarized emission. The integral of $F(x)$ is $\int_0^\infty {\rm d}x\,F(x) = 8\pi/(9\sqrt{3})$, which makes the above spectra consistent with the $L(\alpha)$ function.

Because $G(x) < F(x)$, synchrotron radiation is linearly polarized with polarization degree $\Pi(x) = G(x)/F(x)$. The polarization degree ranges from 50% in the low-frequency limit $\omega \ll \omega_{\rm c}$ to almost 100% in the high-frequency limit $\omega \gg \omega_{\rm c}$.

Radiation from non-thermal particle distribution. Synchrotron radiation is most often associated with ultra-relativistic particles having a broad non-thermal energy distribution. Of particular interest are power-law distributions $N(\gamma) \propto \gamma^{-p}$. The collective synchrotron radiation of such particles has a power-law spectrum $L(\omega) \propto \omega^{-(p-1)/2}$ and polarization degree $\Pi = (p+1)/(p+7/3)$.

9.3 Curvature radiation

It is the radiation of relativistic particles propagating along curved magnetic field lines. The key parameter to determine the acceleration is the curvature radius $R_{\rm c}$, which for velocity v determines acceleration (perpendicular to the velocity) $a_\perp = v^2/R_{\rm c}$. The characteristic radiation frequency is $\omega_{\rm c} = (3/2)\gamma^3(v/R_{\rm c})$, and the total luminosity is $L = (2/3)ce^2(\gamma^4\beta^4/R_{\rm c}^2)$. Compare this with the luminosity of synchrotron radiation, which can be presented in the form $L_{\rm syn}(\alpha) = (2/3)ce^2(\gamma^4\beta^4\sin^4\alpha/R_{\rm L}^2)$. The spectrum of curvature radiation uses the same kernel function $F(x)$.

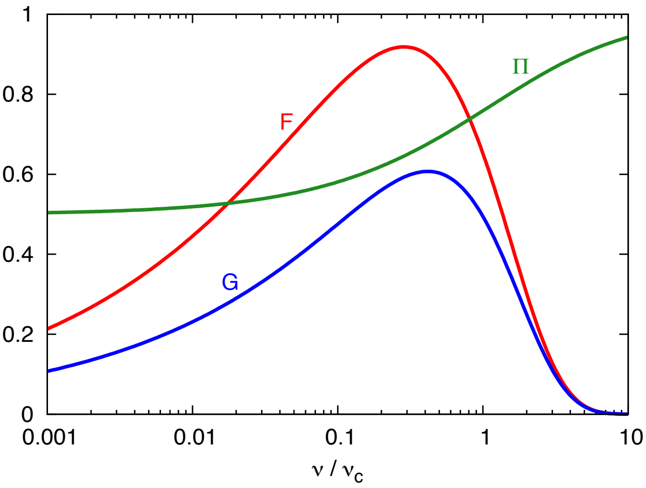

Figure 5: Kernel functions for the total spectrum $F(x)$ (red) and polarized spectrum $G(x)$ (blue) of synchrotron emission, of the argument $x = \omega/\omega_c$ with the characteristic frequency $\omega_c = (3/2)\gamma^3\Omega_L \sin\alpha$. The green line shows the polarization degree $\Pi(x) = G(x)/F(x)$.

10 Radiation scattering off electrons

10.1 Leptonic radiative processes

Of particular interest to astrophysical radiative processes are interactions between radiation and the lightest charged particles – electrons and positrons. Low mass ($\sim$ 2000 times lighter than proton) and elementary structure (as far as one can measure) make electrons more efficient emitters and more sensitive to radiative force. Radiative processes based on electrons (also positrons, muons) are known as *leptonic*.

In the quantum approach, electron interacts with photons. There can be just three basic types of such interaction:

- *scattering*: an electron interacts with a photon, leading to exchange of energy and momentum;
- *annihilation*: an electron interacts with a positron, annihilating into two energetic photons;
- *pair creation*: two energetic photons create a pair of electron and positron.

Note that a free electron cannot emit or absorb a photon, as this would violate conservation of energy and momentum.

10.2 Thomson scattering

The classical approach to the interaction between radiation and a charged particle is known as the *Thomson scattering*.

Consider an electron interacting with a plane electromagnetic wave propagating along $\hat{k} = \hat{z}$. If the electron is initially at rest, it only feels an oscillating electric field. Let the wave be linearly polarized, and consider specifically $E_x(t) = E_0 \cos(\omega t)$. The electron feels a Lorentz force $F_x(t) = -eE_x(t) = -eE_0 \cos(\omega t)$, which causes acceleration $a_x(t) = F_x(t)/m_e = -a_0 \cos(\omega t)$ with $a_0 = eE_0/m_e$. By integration over time, this acceleration is consistent with velocity $v_x(t) = -v_0 \sin(\omega t)$ with $v_0 = eE_0/(m_e\omega)$, and trajectory $x(t) = x_0 \cos(\omega t)$ with $x_0 = eE_0/(m_e\omega^2)$.

Radiation electric field. The accelerated electron is a source of electromagnetic radiation. As long as its motion is non-relativistic, at distant position $\vec{R} = R\hat{R}$ the radiation field is a dipole with an amplitude:

$$E_{\rm rad} \simeq \frac{ea_0}{c^2 R} \sin\Theta = \frac{e^2}{m_e c^2} \frac{E_0}{R} \sin\Theta\,, \tag{10.1}$$

where $\Theta = \angle(\hat{R}, \hat{z})$ is the *scattering angle*. Note that $e^2/(m_e c^2)$ is a length known as the *classical electron radius* $r_e = 2.818 \times 10^{-13}$ cm, hence $E_{\rm rad}/E_0 = (r_e/R)\sin\Theta$.

Total luminosity. For the Larmor's formula, we can use a time average of $a(t)^2$, which is $\left\langle a^2 \right\rangle_t = a_0^2/2$:

$$L = \frac{2e^2}{3c^3} \left\langle a^2 \right\rangle_t = \frac{e^4}{3m_e^2 c^3} E_0^2 = \frac{r_e^2}{3} c E_0^2 = \frac{8\pi r_e^2}{3} \frac{cE_0^2}{8\pi} \equiv \sigma_T \left\langle S_0 \right\rangle_t \,, \tag{10.2}$$

where $\sigma_T = (8\pi/3) r_e^2 \simeq 6.65 \times 10^{-25}\,\mathrm{cm}^2$ is the *Thomson cross section*, and $\left\langle S_0 \right\rangle_t = cE_0^2/8\pi$ is the time-averaged incident Poynting flux density. Thus, in order to sustain emission, the electron absorbs the incident electromagnetic energy as if having a cross section of σ_T.

Polarization. Let us specify the scattered emission direction as $\hat{R} = \hat{z}\cos\Theta + \hat{x}\sin\Theta$. We also need to specify an ortonormal basis including $\hat{R}$. We can choose $\hat{y}' = \hat{y}$ and $\hat{x}' = \hat{y}' \times \hat{R} = \hat{x}\cos\Theta - \hat{z}\sin\Theta$. For an acceleration amplitude $\vec{a} = a_x \hat{x}$, the component perpendicular to $\hat{R}$ is $a_\perp = \vec{a} \cdot \hat{x}' = a_x \cos\Theta$, which implies the time-averaged scattered Poynting flux density

$$S_x = \frac{c}{8\pi} E_{\rm rad}^2 = \frac{e^2 a_x^2}{8\pi c^3 R^2} \cos^2\Theta \,. \tag{10.3}$$

This component is anisotropic in the (x, z) plane, with maxima for $\Theta = 0, 180^\circ$ and no emission for $\Theta = \pm 90^\circ$.

On the other hand, consider a different polarization of the incident wave resulting in acceleration amplitude $\vec{a} = a_y \hat{y}$. The perpendicular component is then $a_\perp = \vec{a} \cdot \hat{y}' = a_y$, and

$$S_y = \frac{e^2 a_y^2}{8\pi c^3 R^2} \,. \tag{10.4}$$

This component is isotropic in the (x, z) plane.

We can identify the total and polarized scattered intensities as:

$$I_{\rm tot} = S_x + S_y = \frac{e^2}{8\pi c^3 R^2} (a_x^2 \cos^2\Theta + a_y^2) \,, \tag{10.5}$$

$$I_{\rm pol} = |S_x - S_y| = \frac{e^2}{8\pi c^3 R^2} |a_x^2 \cos^2\Theta - a_y^2| \,. \tag{10.6}$$

The polarization degree is thus:

$$\Pi = \frac{I_{\rm pol}}{I_{\rm tot}} = \frac{|a_x^2 \cos^2\Theta - a_y^2|}{a_x^2 \cos^2\Theta + a_y^2} \,. \tag{10.7}$$

In the case of unpolarized incident wave, we have $a_x^2 = a_y^2$, and

$$\Pi = \frac{1 - \cos^2\Theta}{1 + \cos^2\Theta} \,. \tag{10.8}$$

10.3 Compton scattering

The quantum approach to interaction between radiation and an electron is known as the *Compton scattering*.

Consider an electron initially at rest interacting with an incident photon of momentum $\vec{p}_1 = (h\nu_1/c)\hat{z}$. The total energy and momentum of this system is:

$$\mathcal{E} = h\nu_1 + m_e c^2 \,, \qquad p_z = \frac{h\nu_1}{c} \,. \tag{10.9}$$

Absorption solution. If one would like to assign this energy and momentum to the electron with recoil velocity $\vec{v}_2 = v_2 \hat{z} \equiv \beta_2 c \hat{z}$, Lorentz factor $\gamma_2 = (1 - \beta_2^2)^{-1/2}$, and dimensionless 4-velocity $u_2 = \gamma_2 \beta_2 = (\gamma_2^2 - 1)^{1/2}$, one would need to satisfy $\mathcal{E} = \gamma_2 m_e c^2$ and $p_z = u_2 m_e c$, which leads to an equation

$$\epsilon_1 \equiv \frac{h\nu_1}{m_e c^2} = \gamma_2 - 1 = u_2 \,, \tag{10.10}$$

which only has a trivial solution $v_2 = 0$ and $\nu_1 = 0$. For a similar reason, a free electron cannot emit a photon.

Scattering solution. The incident photon cannot be absorbed, but it can be scattered at arbitrary angle Θ. Let the scattered photon have a momentum $\vec{p}_2 = (h\nu_2/c)(\hat{z}\cos\Theta + \hat{x}\sin\Theta)$. We also need to account for electron recoil with momentum $\vec{p}_{\rm e,2} = \gamma_2 m_{\rm e} v_2(\hat{z}\cos\Theta_{\rm e} + \hat{x}\sin\Theta_{\rm e})$. Introducing dimensionless photon energy $\epsilon_i = h\nu_i/m_{\rm e}c^2$, the conservation of energy and momentum now involves 3 equations:

$$\frac{\mathcal{E}}{m_{\rm e}c^2} = \epsilon_1 + 1 = \epsilon_2 + \gamma_2\,, \tag{10.11}$$

$$\frac{p_z}{m_{\rm e}c} = \epsilon_1 = \epsilon_2\cos\Theta + u_2\cos\Theta_{\rm e}\,, \tag{10.12}$$

$$\frac{p_x}{m_{\rm e}c} = 0 = \epsilon_2\sin\Theta + u_2\sin\Theta_{\rm e}\,. \tag{10.13}$$

The 3 unknowns are ϵ_2, v_2 (or γ_2 or u_2), and $\Theta_{\rm e}$. The electron scattering angle can be eliminated from the x-momentum as $\sin\Theta_{\rm e} = -(\epsilon_2/u_2)\sin\Theta$. Substituting to the z-momentum yields $u_2^2 = \epsilon_1^2 + \epsilon_2^2 - 2\epsilon_1\epsilon_2\cos\Theta$. On the other hand, the energy equation gives $\gamma_2^2 = (\epsilon_1 - \epsilon_2 + 1)^2$. After eliminating u_2, the solution for scattered photon energy is:

$$\epsilon_2 = \frac{\epsilon_1}{1 + \epsilon_1(1 - \cos\Theta)}\,. \tag{10.14}$$

Thomson and Klein-Nishina regimes. The energy change $\epsilon_2 - \epsilon_1$ is negligible for $\epsilon_1 \ll 1$ or $h\nu_1 \ll m_{\rm e}c^2$ (the classical limit corresponding to the Thomson scattering). Once the incident photon energy becomes comparable to the electron rest energy of $m_{\rm e}c^2 = 511$ keV (soft gamma rays), its energy will be reduced upon scattering ($\epsilon_2 \le \epsilon_1$). For head-on scattering ($\Theta = 180^\circ$), $\epsilon_2 = \epsilon_1/(1+2\epsilon_1)$.

An energetic gamma-ray photon with $\epsilon_1 \gg 1$ can in principle deposit most of its energy to the electron, but it would also be increasingly more likely to produce electron-positron pairs. This is known as the *Klein-Nishina regime*, in which the scattering cross section $\sigma_{\rm KN}$ becomes systematically lower than $\sigma_{\rm T}$.

10.4 Inverse Compton scattering

Astrophysical electrons are often highly energetic, even ultra-relativistic with Lorentz factors $\gamma \gg 1$. We have direct evidence for this from detections of cosmic rays (they are just a minor ingredient, but very important at $\sim$ GeV energies) and solar energetic particles. Considered in the electron's rest frame, this process is exactly the same Compton scattering, however, to understand the results in the frame of astrophysical interest (could be a source frame like AGN black hole, but also the co-moving frame of a relativistic jet), both the incident and scattered photons need to be Lorentz-transformed.

Reference frames. In a reference frame $\mathcal{O}$, consider an ultra-relativistic electron with $\gamma = (1 - \beta^2)^{-1/2} \gg 1$ propagating along $\hat{z}$. The electron rest frame is denoted as $\mathcal{O}'$.

Incident photon. Consider an incident photon, which in $\mathcal{O}$ has momentum $\vec{p}_1 = p_1(\hat{z}\cos\theta_1 + \hat{x}\sin\theta_1)$ with $p_1 = \mathcal{E}_1/c = h\nu_1/c$. In $\mathcal{O}'$, the momentum of this photon is likewise $\vec{p}_1' = p_1'(\hat{z}\cos\theta_1' + \hat{x}\sin\theta_1')$. The parameters p_1' and $\mu_1' = \cos\theta_1'$ are related to p_1 and $\mu_1 = \cos\theta_1$ by Lorentz transformation:

$$p_1' = \gamma(1 - \beta\mu_1)p_1 \equiv \mathcal{D}_1 p_1\,, \qquad \mu_1' = \frac{\mu_1 - \beta}{1 - \beta\mu_1}\,, \tag{10.15}$$

where a Doppler factor $\mathcal{D}_1$ has been introduced. Those are the *relativistic Doppler and aberration effects*, respectively. The aberration law implies also that $\mathrm{d}\Omega_1' = (\mathrm{d}\mu_1'/\mathrm{d}\mu_1)\,\mathrm{d}\Omega_1 = \mathrm{d}\Omega_1/\mathcal{D}_1^2$.

Scattered photon. Now consider a scattered photon in $\mathcal{O}'$. For simplicity, let us confine the problem to the $(\hat{x}, \hat{z})$ plane, so that the scattered photon momentum is $\vec{p}_2' = p_2'(\hat{z}\cos\theta_2' + \hat{x}\sin\theta_2')$. This photon is transformed to $\mathcal{O}$ using reversed Lorentz transformation:

$$p_2 = \gamma(1 + \beta\mu_2')p_2'\,, \qquad \mu_2 = \frac{\mu_2' + \beta}{1 + \beta\mu_2'}\,. \tag{10.16}$$

Example. As a typical example, consider in $\mathcal{O}$ an incident photon with $\theta_1 \sim 90^\circ$, so that $\mu_1 \sim 0$. In $\mathcal{O}'$, relativistic aberration results in $\mu_1' \simeq -\beta$, hence $\theta_1' \simeq 180^\circ - 1/\gamma$; and the photon momentum is relativistically boosted $p_1' \simeq \gamma p_1$. For scattering in the Thomson regime, $p_2' \simeq p_1'$. For the scattering angle, let us take $\Theta' \sim 90^\circ$. Then, the scattered photon would have $\theta_2' = \theta_1' \pm \Theta' \sim \pm 90^\circ$, hence $\mu_2' \sim 0$. Transformation back into $\mathcal{O}$ yields $p_2 \simeq \gamma p_2'$ and $\mu_2 \simeq \beta$ or $\theta_2 \simeq 1/\gamma$, closely along the electron velocity. The combined momentum boost is $p_2 \simeq \gamma p_2' \simeq \gamma p_1' \simeq \gamma^2 p_1$.

Luminosity. The power of radiation scattered off a single relativistic electron can be first evaluated in its rest frame $\mathcal{O}'$. In the Thomson regime, the energy of scattered photons does not change, hence the power of scattered radiation equals the power of incident radiation. The number of incident photons interacting with an electron per unit time $\mathrm{d}t'$ is $\mathrm{d}N_1' = n_1'\,\mathrm{d}V' = n_1'\sigma_\mathrm{T} c\,\mathrm{d}t'$, where n_1' is their number density. This scaling is independent of the photon energy and direction (momentum). However, the amount of incident energy depends on the photon energy distribution:

$$\frac{\mathrm{d}\mathcal{E}_1'}{\mathrm{d}\epsilon_1'} = \epsilon_1' m_\mathrm{e} c^2 \frac{\mathrm{d}N_1'}{\mathrm{d}\epsilon_1'} = c\sigma_\mathrm{T}\, m_\mathrm{e} c^2\, \epsilon_1' \frac{\mathrm{d}n_1'}{\mathrm{d}\epsilon_1'}\,\mathrm{d}t' . \tag{10.17}$$

Recall that the phase space distribution of photons $f = \mathrm{d}N/(\mathrm{d}^3\vec{r}\,\mathrm{d}^3\vec{p}) = \mathrm{d}n/\mathrm{d}^3\vec{p}$ is Lorentz invariant. This means that $\mathrm{d}n/(\epsilon^2\,\mathrm{d}\epsilon\,\mathrm{d}\Omega)$ is Lorentz invariant. Since $\epsilon'/\epsilon = \mathcal{D}(\mu)$ and $\mathrm{d}\Omega'/\mathrm{d}\Omega = 1/\mathcal{D}(\mu)^2$, then $\mathrm{d}n/\mathrm{d}\epsilon$ is also Lorentz invariant. One can now write:

$$\frac{\mathrm{d}\mathcal{E}_1'}{\mathrm{d}t'} = c\sigma_\mathrm{T}\, m_\mathrm{e} c^2 \int \mathrm{d}\epsilon_1'\, \epsilon_1' \frac{\mathrm{d}n_1}{\mathrm{d}\epsilon_1} . \tag{10.18}$$

Consider that the incident radiation is unidirectional with fixed μ_1, then one can write that $\mathrm{d}\epsilon_1' = \mathcal{D}_1(\mu_1)\,\mathrm{d}\epsilon_1$, so that:

$$\frac{\mathrm{d}\mathcal{E}_1'}{\mathrm{d}t'}(\mu_1) = c\sigma_\mathrm{T}\mathcal{D}_1(\mu_1)^2 \left[m_\mathrm{e} c^2 \int \mathrm{d}\epsilon_1\, \epsilon_1 \frac{\mathrm{d}n_1(\mu_1)}{\mathrm{d}\epsilon_1}\right] \equiv c\sigma_\mathrm{T}\gamma^2(1-\beta\mu_1)^2\, u_\mathrm{rad}(\mu_1) , \tag{10.19}$$

where we introduce the radiation energy density $u_\mathrm{rad}(\mu_1)$. The average value of the explicitly anisotropic term is $\left\langle(1-\beta\mu_1)^2\right\rangle_{\Omega_1} = 1 + \beta^2/3 \simeq 4/3$. Hence, in the case of incident radiation isotropic in $\mathcal{O}$, i.e., u_rad independent of μ_1, once has

$$L_\mathrm{IC,iso} = \left\langle \frac{\mathrm{d}\mathcal{E}_1'}{\mathrm{d}t'} \right\rangle_{\Omega_1} = \frac{4}{3} c\sigma_\mathrm{T}\gamma^2 u_\mathrm{rad} , \tag{10.20}$$

Analogy with the synchrotron radiation. Recall that for the synchrotron radiation, the pitch-angle averaged luminosity for an ultra-relativistic electron is $L_\mathrm{syn,iso} \simeq (4/3)c\sigma_\mathrm{T}\gamma^2 u_\mathrm{B}$. This formula is strikingly similar to the one for inverse Compton luminosity, with magnetic energy density u_B playing the role of incident radiation energy density u_rad. This similarity can be used directly in the studies of blazars, in which the same electrons can produce synchrotron radiation to explain the low-energy non-thermal spectral component and upscatter various radiation fields by the inverse Compton process to explain the high-energy spectral component. The luminosity ratio (Compton dominance) can be used to constrain relative energy densities in the emitting region (within a relativistic jet frame $\mathcal{O}'$):

$$\frac{L_\mathrm{IC}}{L_\mathrm{syn}} = \frac{u_\mathrm{rad}'}{u_\mathrm{B}'} . \tag{10.21}$$

Quantum picture of the synchrotron radiation. This suggests a deeper similarity between these two processes. In the QED, synchrotron emission can be interpreted as the inverse Compton scattering of virtual photons associated with the magnetic field. The characteristic frequency of the synchrotron radiation, $\omega_\mathrm{c} \sim \gamma^2\Omega_\mathrm{c}$ with $\Omega_\mathrm{c} = eB/m_\mathrm{e}c$ the cyclotron frequency (non-relativistic Larmor frequency), suggests that those photons have energies $\sim \hbar\Omega_\mathrm{c}$. One can also evaluate that synchrotron radiation is equivalent to the Thomson scattering, unless $\gamma\hbar\Omega_\mathrm{c} \sim m_\mathrm{e}c^2$. Without the γ factor, this condition defines the critical magnetic field strength $B_\mathrm{cr} = m_\mathrm{e}^2 c^3/(\hbar e) \simeq 4.4\times 10^{13}\,\mathrm{G}$, which is exceeded in the magnetars.

11 Radiation propagating through plasmas

When an electromagnetic wave propagates through a plasma (e.g., interstellar medium), charged particles are accelerated by the radiation electric field. This classical description is similar to the problem of Thomson scattering, but now we consider many charges. Once again, electrons are most affected due to their low mass.

11.1 Dispersion

Consider a linearly polarized wave propagating along $\hat{k} = \hat{z}$ with electric field $E_x(t,z) = E_1 \exp(i\omega t + ikz)$. This electric field causes electron acceleration $a_x(t,z) = -eE_x/m_\mathrm{e}$ and velocity $v_x(t,z) = ieE_x/(\omega m_\mathrm{e})$, out of phase with $a_x(t,z)$. Motions of electrons of number density n_e contribute to electric current $j_x(t,z) = -en_\mathrm{e}v_x = -ie^2 n_\mathrm{e} E_x/(\omega m_\mathrm{e})$. This electric current contributes to the x component of the Ampere's law:

$$\begin{aligned} c(\vec{\nabla} \times \vec{B})_x &= 4\pi j_x + \partial_t E_x\,, && (11.1) \\ -ickB_y &= -4\pi \frac{ie^2 n_\mathrm{e} E_x}{\omega m_\mathrm{e}} + i\omega E_x\,, && (11.2) \\ -ckB_y &= \left(-\frac{4\pi e^2 n_\mathrm{e}/m_\mathrm{e}}{\omega^2} + 1\right)\omega E_x\,. && (11.3) \end{aligned}$$

The term $4\pi e^2 n_\mathrm{e}/m_\mathrm{e} \equiv \omega_\mathrm{p}^2$ is the squared *plasma frequency*, a function only of electron density: $\omega_\mathrm{p} \simeq 60\, n_{\mathrm{e},0}^{1/2}$ kHz. The magnetic field component can be substituted from the Faraday's law $B_y = -(kc/\omega)E_x$ to obtain a dispersion relation:

$$\omega^2 = k^2c^2 + \omega_\mathrm{p}^2\,. \tag{11.4}$$

Note the following implications:

- The phase speed is $v_\mathrm{ph} = \omega/k \equiv c/n_\mathrm{r} > c$, where

$$n_\mathrm{r} = \left[1 + \left(\frac{\omega_\mathrm{p}}{kc}\right)^2\right]^{-1/2} = \left[1 - \left(\frac{\omega_\mathrm{p}}{\omega}\right)^2\right]^{1/2} < 1 \tag{11.5}$$

  is the *index of refraction*.

- The group speed is $v_\mathrm{gr} = \partial\omega/\partial k = cn_\mathrm{r} < c$. Information (e.g., pulses of pulsars, fast radio bursts) travelling with the group speed will be *dispersed* and delayed by

$$\Delta t(\omega) = \int_R \frac{\mathrm{d}r}{v_\mathrm{gr}(\omega)} - \frac{R}{c} = \frac{1}{c}\int_R \frac{\mathrm{d}r}{n_\mathrm{r}(\omega)} - \frac{R}{c} \simeq \frac{1}{2c\omega^2}\int_R \mathrm{d}r\omega_\mathrm{p}^2 = \frac{2\pi e^2}{m_\mathrm{e}c\omega^2}\int_R \mathrm{d}r\, n_\mathrm{e} \equiv \frac{2\pi e^2}{m_\mathrm{e}c\omega^2}\mathrm{DM}\,, \tag{11.6}$$

  with the *dispersion measure* $\mathrm{DM} \equiv \int_R \mathrm{d}r\, n_\mathrm{e}$; the linear approximation made in the limit of $\omega \gg \omega_\mathrm{p}$.

- For $\omega < \omega_\mathrm{p}$, k becomes imaginary, which means that electromagnetic wave is exponentially damped on length scale $c/(\omega_\mathrm{p}^2 - \omega^2)^{1/2}$. In the limit of $\omega \ll \omega_\mathrm{p}$, this length scale is known as the *skin depth* $d_\mathrm{p} = c/\omega_\mathrm{p}$.

11.2 Faraday Rotation

Let us extend the above problem by including uniform magnetic field along the wavevector: $\vec{B}_0 = B_0\hat{k} = B_0\hat{z}$. The wave electric field can be parametrized by amplitude and polarization angle $\vec{E}_1 = E_1(\hat{x}\cos\chi + \hat{y}\sin\chi) \equiv E_1\exp(i\chi)$, either of which may have a time dependence of $\propto \exp(i\omega t)$. The electron velocity $\vec{\beta}$ is now governed by the Lorentz force $\vec{F}_\mathrm{L} = -e(\vec{E}_1 + \vec{\beta} \times \vec{B}_0) = m_\mathrm{e}\vec{a} = i\omega m_\mathrm{e}c\vec{\beta}$. Noting that $\vec{\beta} \times \vec{B}_0 \equiv -iB_0\vec{\beta}$, one obtains:

$$-\frac{\vec{E}_1}{B_0} + i\vec{\beta} = \frac{\omega}{\Omega_\mathrm{c}} i\vec{\beta}\,, \tag{11.7}$$

where the cyclotron frequency $\Omega_\mathrm{c} = eB_0/(m_\mathrm{e}c)$ was substituted.

The solution is:

$$\vec{\beta} = \frac{\Omega_\mathrm{c}/\omega}{1-\Omega_\mathrm{c}/\omega}\frac{i\vec{E}_1}{B_0}\,, \tag{11.8}$$

where we anticipate that $|\omega/\Omega_\mathrm{c}| > 1$.

Note that different signs of B_0 or ω result in solutions of two different amplitudes. Fixing $B_0 > 0$ and hence $\Omega_\mathrm{c} > 0$, two signs of ω correspond to the 2 fundamental circular polarizations. Note that ω combines the time dependences of amplitude E_1 and polarization angle χ. For circular polarization, E_1 is constant and $\chi = \chi_0 + \omega t$. For linear polarization, χ is constant and $E_1 \propto \exp(i\omega t)$, which is nevertheless equivalent to a combination of two opposite circular polarizations.

We now need to combine the Ampere's law with the Faraday equation (note that $\vec{\nabla}\times\vec{B}_1 = ik(\hat{z}\times\vec{B}_1) \equiv ik(-i)\vec{B}_1 = k\vec{B}_1$, and likewise $\vec{\nabla}\times\vec{E}_1 \equiv k\vec{E}_1$):

$$\begin{aligned} kc\vec{B}_1 &= 4\pi\vec{j} + i\omega\vec{E}_1\,, & (11.9)\\ kc\vec{E}_1 &= -i\omega\vec{B}_1\,, & (11.10)\\ (k^2c^2-\omega^2)i\vec{E}_1 &= 4\pi\omega\vec{j}\,, & (11.11) \end{aligned}$$

Substituting $\vec{j} = -cen_\mathrm{e}\vec{\beta}$:

$$(k^2c^2-\omega^2)\frac{i\vec{E}_1}{B_0} = -\omega\frac{4\pi e^2 n_\mathrm{e}}{m_\mathrm{e}}\frac{m_\mathrm{e}c}{eB_0}\vec{\beta} = -\omega_\mathrm{p}^2\frac{\omega}{\Omega_\mathrm{c}}\vec{\beta}\,. \tag{11.12}$$

Substituting $\vec{\beta}$, we obtain a dispersion relation and expression for phase speed $v_\mathrm{ph} = \omega/k$:

$$\begin{aligned} k^2c^2-\omega^2 &= -\frac{\omega_\mathrm{p}^2}{1-\Omega_\mathrm{c}/\omega}\,, & (11.13)\\ (c/v_\mathrm{ph})^2 &= 1-\frac{(\omega_\mathrm{p}/\omega)^2}{1-\Omega_\mathrm{c}/\omega}\,. & (11.14) \end{aligned}$$

We now consider a limit of $\omega \gg \omega_\mathrm{p}, \Omega_\mathrm{c}$. The linearized phase speeds for $\omega = \pm\omega_0$ are:

$$\frac{v_{\mathrm{ph},\pm}}{c} \simeq 1 + \frac{\omega_\mathrm{p}^2}{2\omega_0^2} + \frac{\Omega_\mathrm{c}^2}{2\omega_0^2} \mp \frac{\omega_\mathrm{p}^2\Omega_\mathrm{c}}{2\omega_0^3} + ... \tag{11.15}$$

The phase speed difference is

$$\frac{\Delta v_\mathrm{ph}}{c} \simeq \frac{\omega_\mathrm{p}^2\Omega_\mathrm{c}}{\omega_0^3}\,. \tag{11.16}$$

Over distance Δz this results in a phase difference (substituting $k_0 = \omega_0/c = 2\pi/\lambda_0$):

$$\Delta\phi = \frac{\Delta v_\mathrm{ph}}{c}k_0\,\Delta z \simeq \frac{\omega_\mathrm{p}^2\Omega_\mathrm{c}}{c\omega_0^2}\,\Delta z = \frac{4\pi e^3}{m_\mathrm{e}^2c^2(2\pi c)^2}\,\lambda_0^2\,n_\mathrm{e}B_0\,\Delta z\,. \tag{11.17}$$

This phase difference corresponds to twice the rotation of the polarization angle:

$$\Delta\chi = \frac{\Delta\phi}{2} = \lambda_0^2\,\frac{e^3}{2\pi m_\mathrm{e}^2c^4}\,n_\mathrm{e}B_0\,\Delta z \equiv \lambda_0^2\,\mathrm{RM}\,, \tag{11.18}$$

where RM stands for the *rotation measure*. Over long astrophysical distances r, electron density n_e and magnetic field component $B_\parallel$ parallel to the line of sight (and positive towards the observer) may be functions of r. Hence, astrophysical rotation measures are line-of-sight integrals:

$$\mathrm{RM} = \frac{e^3}{2\pi m_\mathrm{e}^2c^4}\int_R \mathrm{d}r\,n_\mathrm{e}\,B_\parallel\,. \tag{11.19}$$

For uniform distributions of RM (Faraday screens), rotation of polarization vectors scales with λ^2.

Faraday rotation is a very important effect in radio astronomy. On one hand, it makes difficult measuring the true polarization of radio sources; on the other hand, it allows to estimate the strengths and distributions of magnetic fields in extended sources, especially across the Milky Way.

12 Absorption processes

Each emission process is associated with an absorption process. In particular, bremsstrahlung is associated with the *free-free absorption*, and synchrotron radiation is associated with the *synchrotron self-absorption*. In general, absorption becomes important at low frequencies, at which the individual processes of photon emission (spontaneous or stimulated) and photon absorption become more probable and more balanced. Below certain characteristic frequency, a particular source may become optically thick to a particular absorption process. In such conditions, a thermodynamic equilibium can be achieved, at least between photons of given frequency and electrons of corresponding characteristic energy. In the case of free-free emission, the optically thick spectrum hardens to $F(\nu) \propto \nu^2$, consistent with the Rayleigh-Jeans thermal spectrum. In the case of synchrotron self-absorption, the optically thick spectrum hardens to $F(\nu) \propto \nu^{5/2}$.

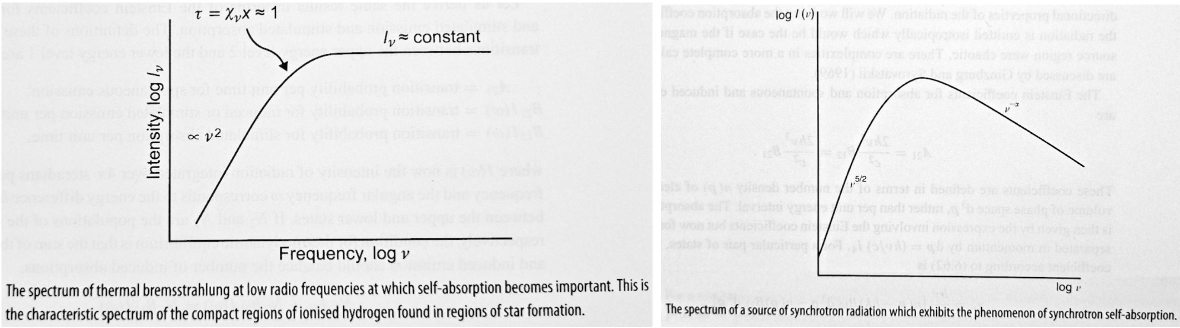

Figure 6: Left: schematic spectrum of the bremsstrahlung emission with free-free absorption (Fig 6.4 in Longair, 2003). Right: schematic spectrum of the synchrotron emission with synchrotron self-absorption at low frequencies (Fig 8.12 in Longair, 2003).

13 Pair production and annihilation

Basic interactions between electrons/positrons and photons (interaction of two particles resulting in two particles) include: (1) Compton scattering, (2) photon-photon production of electron-positron pair, and (3) annihilation of electron and positron into a pair of photons. Pair production and annihilation involve gamma-ray photons with energy exceeding the rest energy of electron/positron $h\nu \ge m_{\rm e}c^2$. Positrons can also be produced in other processes, including β decays and pair production in strong electromagnetic fields. Positrons are the most common form of antimatter in the Universe.

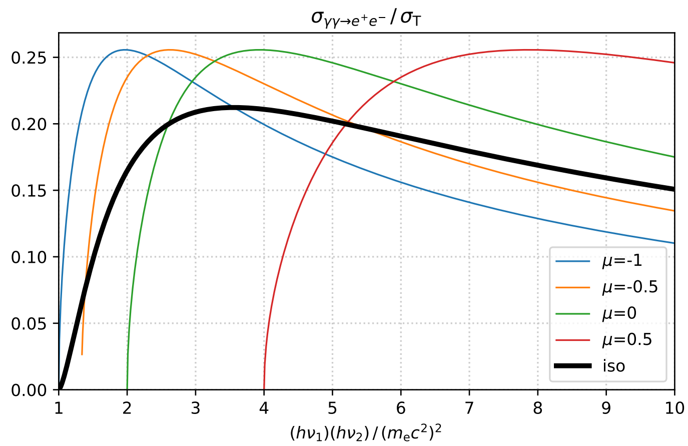

Figure 7: Cross section for the photon-photon pair production process, normalized to the Thomson cross section. Thin lines are for fixed cosine $\mu = \cos(\theta)$ of the angle θ between incident photons. The thick line shows the average cross section for isotropic distribution of target photons.

14 Hadronic processes

Radiation is produced by accelerating charged particles, this course has been focused on the electrons (and positrons) as the lightest stable abundant charges. However, heavier charges can also contribute to the cosmic radiation. Protons (*baryons*) and other atomic nuclei are abundant and stable, can be accelerated to extremely high energies ($\sim 10^{20}$ eV), making up most of the cosmic rays. They can produce radiation by acceleration in magnetic fields (proton synchrotron) or by colliding with other particles, which results in cascades (showers) of secondary particles including leptons, unstable *mesons* (mostly charged or neutral pions), which in turn produce radiation. Baryons are mesons are collectively known as the *hadrons*, hence the overall term *hadronic processes*.

14.1 Proton synchrotron

Let us compare a relativistic proton with Lorentz factor $\gamma_{\rm p}$ with an electron with the same energy $\mathcal{E}_{\rm e} = \gamma_{\rm e} m_{\rm e} c^2 = \gamma_{\rm p} m_{\rm p} c^2 = \mathcal{E}_{\rm p}$. Such electron is even more relativistic with $\gamma_{\rm e} = (m_{\rm p}/m_{\rm e})\gamma_{\rm p}$. The Larmor frequency of the proton $\Omega_{\rm L,p} = eB_0/(\gamma_{\rm p} m_{\rm p} c)$ is exactly the same as for the electron. Acceleration in the reference frame of interest, perpendicular to the velocity vector, is $a_{\rm p,\perp} = \Omega_{\rm L,p} v_{\rm p,\perp} \sim \Omega_{\rm L,p} c$, comparable to that for the electron. However, acceleration in the rest frame of the proton is $a'_{\rm p} = \gamma_{\rm p}^2 a_{\rm p,\perp}$. In the Larmor's formula for total luminosity, we use squared rest-frame acceleration $L_{\rm syn,p} = (2/3)(e^2/c^3)\gamma_{\rm p}^4 \Omega_{\rm L,p}^2 v_{\rm p,\perp}^2$. Hence, for relativistic proton and electron of equal energy, the ratio of synchrotron luminosities is $L_{\rm syn,p}/L_{\rm syn,e} \simeq (\gamma_{\rm p}/\gamma_{\rm e})^4 = (m_{\rm p}/m_{\rm e})^{-4}$. Correspondingly, the synchrotron cooling time scale for a proton will be longer by factor $(m_{\rm p}/m_{\rm e})^4$, and its characteristic frequency will be $\omega_{\rm c,p}/\omega_{\rm c,e} = (\gamma_{\rm p}/\gamma_{\rm e})^3 = (m_{\rm p}/m_{\rm e})^{-3}$. Because the luminosity per particle is so much lower, proton synchrotron process is much less efficient energetically than electron synchrotron. This is generally true for all hadronic processes. Nevertheless, in the presence of extremely energetic protons they may be important and are being considered. An important motivation is to explain observations of highly energetic neutrinos (e.g. by the IceCube experiment) inferred to have an astrophysical origin.