PROBLEM 7: MAGNETOCENTRIFUGE

- Consider an axisymmetric poloidal magnetic field line in cylindrical coordinates (r, ϕ, z): $\overrightarrow{B} = B[\cos\theta, 0, \sin\theta]$ crossing the $z = 0$ plane (accretion disk) at $r = r_0$, rotating with Keplerian angular velocity $\Omega^2 = GM/r_0^3$ in the gravitational field of a compact object of mass M located at [0,0,0].

- Calculate the net force acting on a charged particle of mass m tied to this magnetic field line, a combination of gravitational and centrifugal forces projected along $\overrightarrow{B}$, as function of z.

- What is the condition for this net force to point away from the accretion disk?

This problem is worth 5 points. Solutions should be sent as 1-page PDF files to knalew@camk.edu.pl before the next lecture.

NET FORCE ALONG $\hat{B}$

- Our particle is located at $[r,0,z]$ that satisfies $r(z) = r_0 + \frac{z}{\tan\theta}$.

- The gravitational force is $\overrightarrow{F}_g = \frac{GMm}{R^3}[-r,0,-z]$, where $R(z) = \sqrt{r^2 + z^2}$.

- The centrifugal force is $\overrightarrow{F}_\Omega = m\Omega^2[r,0,0] = \frac{GMm}{r_0^3}[r,0,0]$.

- The net force is $\overrightarrow{F} = \overrightarrow{F}_g + \overrightarrow{F}_\Omega = \frac{GMm}{r_0^3}\left[\left(1 - \frac{r_0^3}{R^3}\right)r,0,-\frac{r_0^3}{R^3}z\right]$.

- The net force along $\hat{B} = [\cos\theta, 0, \sin\theta]$:

$$\overrightarrow{F}\cdot\hat{B} = \frac{GMm}{r_0^3}\left[r\cos\theta - \frac{r_0^3}{R^3}\left(r_0\cos\theta + \frac{z}{\sin\theta}\right)\right].$$

SOLUTIONS

- $\overrightarrow{F} \cdot \hat{B} = \frac{GMm}{r_0^3}\left[r\cos\theta - \frac{r_0^3}{R^3}\left(r_0\cos\theta + \frac{z}{\sin\theta}\right)\right]$

- In the limit $z \ll r_0$: $R \simeq r$

  and $\overrightarrow{F} \cdot \hat{B} \simeq \frac{GMm}{r_0^3}\frac{(4\cos^2\theta - 1)}{\sin\theta}z.$

- For $\sin\theta > 0$:

  $\overrightarrow{F} \cdot \hat{B} > 0$ for $|\cos\theta| > \frac{1}{2}$,

  hence $\theta < 60^\circ$ or $\theta > 120^\circ$.