Analytic Models of the Kerr Black Hole Accretion Discs:

A Minimalistic Picture

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Spacetime Geometry — Matter

Spacetime geometry g_{ik} fixed	Matter $\nabla_i T^i{}_k = 0$

- Symmetries
- Observers
- Geodesic motion
- Inertial forces
- Circular geodesics

- $T^i{}_k$ = fluid + stress
- Non-inertial forces
- Dissipation
- Radiation
- Observational properties

The Kerr Metric

(M = mass, a = spin)

$$\text{signature } +---, \qquad c = 1 = G$$

$$g_{tt} = \left(1 - \frac{2Mr}{\Sigma^2}\right), \qquad g_{t\varphi} = \frac{2Mar\sin^2\theta}{\Sigma^2}$$

$$g_{\varphi\varphi} = -\left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma^2}\right)\sin^2\theta$$

$$g_{rr} = -\frac{\Sigma^2}{\Delta}, \qquad g_{\theta\theta} = -\Sigma^2$$

$$\Delta = r^2 - 2Mr + a^2, \qquad \Sigma^2 = r^2 + a^2\cos^2\theta$$

The Schwarzschild Metric

$(a = 0)$

$$g_{tt} = \left(1 - \frac{r_H}{r}\right), \qquad r_H = \frac{2GM}{c^2}$$

$$g_{rr} = -\left(1 - \frac{r_H}{r}\right)^{-1}$$

$$g_{\varphi\varphi} = -r^2 \sin^2\theta, \qquad g_{\theta\theta} = -r^2$$

Symmetries: Killing Vectors

Time symmetry $\eta^i = \delta^i_t, \quad \partial_t g_{ik} = 0$

Axial symmetry $\xi^i = \delta^i_\varphi, \quad \partial_\varphi g_{ik} = 0$

$$x^i = (\eta^i, \xi^i), \qquad y^i = (\eta^i, \xi^i)$$

$$\nabla_i x_k + \nabla_k x_i = 0, \qquad x^i y_i \equiv (xy)$$

$$x^i \nabla_i y_k \; = \; y^i \nabla_i x_k \; = \; -\tfrac{1}{2} \nabla_k (xy)$$

$\nabla_i =$ covariant derivative

Geodesic Motion

$$u^i = \frac{dx^i}{ds} = \text{four velocity}, \qquad a_k = u^i \nabla_i u_k = \text{four acceleration}$$

$$a_k = 0 \quad \text{(geodesic motion)}$$

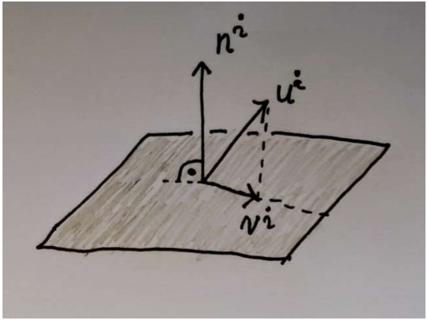
if $a_k = 0$ and x^i = Killing vector, then

$$u^i \nabla_i (ux) = 0 \implies (ux) \text{ conserved along geodesic line}$$

$$\mathcal{E} = (u\eta) = \text{energy}$$

$$\mathcal{L} = -(u\xi) = \text{angular momentum}, \qquad \ell = \frac{\mathcal{L}}{\mathcal{E}} = \text{specific angular momentum}$$

The Observer and 3-Velocity



n^i = observer

u^i = four-velocity

v^i = 3-velocity

$$v^i = V\tau^i$$

V = speed, τ^i = direction, $(\tau\tau) = -1$

$$u^i = \gamma\left(n^i + V\tau^i\right) \quad \text{unique decomposition}, \qquad \gamma = \frac{1}{\sqrt{1 - V^2}}$$

Special Relativity in a Nutshell

$$u^i = \gamma(n^i + v\tau^i) \quad \text{first observer}$$

$$u^i = \Gamma(N^i + V\mathcal{T}^i) \quad \text{second observer}$$

$$\left.\begin{aligned} N^i &= \gamma_0(n^i + v_0\tau_0^i) \\ \mathcal{T}_0^i &= \gamma_0(v_0 n^i + \tau_0^i) \end{aligned}\right\} \text{Lorentz transformations}$$

$$(NN) = 1, \qquad (\tau_0\tau_0) = -1, \qquad (N\tau_0) = 0$$

$$\boxed{V = \frac{v + v_0}{1 + vv_0}}$$

Steady Circular Motion

$$u^i = \alpha\eta^i + \beta\xi^i, \qquad \eta^i\nabla_i\alpha = 0 = \xi^i\nabla_i\alpha, \qquad \eta^i\nabla_i\beta = 0 = \xi^i\nabla_i\beta$$

$$= A(\eta^i + \Omega\xi^i), \qquad A^{-2} = (\eta\eta) + 2\Omega(\eta\xi) + \Omega^2(\xi\xi)$$

NOTE: η^i is not a unit vector

$$\Omega = \frac{u^\varphi}{u^t} = \text{angular velocity}$$

$$\Omega = -\frac{\ell(\eta\eta) + (\eta\xi)}{\ell(\eta\xi) + (\xi\xi)}, \qquad \text{Schwarzschild: } \tilde{r}^2 = -\frac{(\xi\xi)}{(\eta\eta)} = \text{gyration radius}$$

$$\Omega = -\ell\frac{(\eta\eta)}{(\xi\xi)} = \ell\,\tilde{r}^{-2}$$

The ZAMO Observer

Note: $(n\xi) = \ell_{ZAMO} = 0$

$$n^i = e^{\Phi}\left(\eta^i + \omega\xi^i\right) \quad \text{unique!}$$

$$\Phi = -\frac{1}{2}\ln\left[(\eta\eta) + \omega(\eta\xi)\right] = \text{gravitational potential}$$

$$\omega = -\frac{(\eta\xi)}{(\xi\xi)} = \text{dragging of inertial frames}$$

Acceleration in Circular Motion

$$0 = a_k = u^i \nabla_i u_k = A(\eta^i + \Omega \xi^i) \nabla_i \left[A(u_k + \Omega \xi_k) \right]$$

Because $\eta^i \nabla_i \Omega = \xi^i \nabla_i \Omega = \eta^i \nabla_i A = \xi^i \nabla_i A = 0$

$$\begin{aligned} 0 = a_k &= A^2 \left\{ \eta^i \nabla_i \eta_k + 2\Omega \eta^i \nabla_i \xi_k + \Omega^2 \xi^i \nabla_i \xi_k \right\} \\ &= -\tfrac{1}{2} A^2 \left[\partial_k (\eta\eta) + 2\Omega \partial_k (\eta\xi) + \Omega^2 \partial_k (\xi\xi) \right] \end{aligned}$$

Finally: $\quad \partial_r (\eta\eta) + 2\Omega \partial_r (\eta\xi) + \Omega^2 \partial_r (\xi\xi) = 0$

$(\eta\eta) = g_{tt}$, $(\eta\xi) = g_{t\varphi}$, $(\xi\xi) = g_{\varphi\varphi}$ — these are known functions

Quadratic equation for Ω. In Schwarzschild

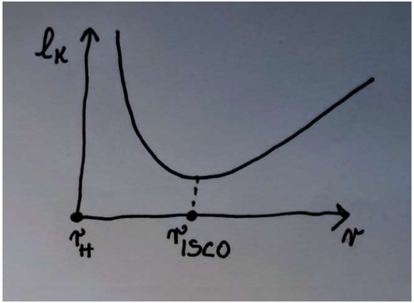
$$\Omega = \Omega_K = \left(\frac{GM}{r^3} \right)^{1/2} \qquad \textbf{KEPLER!!}$$

Keplerian Motion

$$\ell_K = \Omega_K \tilde{r}^2, \qquad \tilde{r}^2 = \frac{r^2}{1 - r_H/r}$$

$$= \left(\frac{GM}{r^3}\right)^{1/2} \frac{r^2}{1 - r_H/r} = \frac{(GMr)^{1/2}}{1 - r_H/r}$$

$$\ell_K(\infty) = \ell_K(r_H) = \infty$$



therefore ℓ_K has a minimum between ∞ and r_H

The Effective Potential

$$\partial_r(\eta\eta) + 2\Omega\partial_r(\eta\xi) + \Omega^2\partial_r(\xi\xi) = 0$$

$$\Omega = -\frac{\ell(\eta\eta) + (\eta\xi)}{\ell(\eta\xi) + (\xi\xi)} \qquad \Downarrow \quad \text{easy but long algebra}$$

$$\partial_r g^{tt} + 2\ell\, \partial_r g^{t\varphi} + \ell^2 \partial_r g^{\varphi\varphi} = 0$$

$$\Updownarrow$$

$$E_{\text{eff}} = g^{tt} - 2\ell g^{t\varphi} + \ell^2 g^{\varphi\varphi}$$

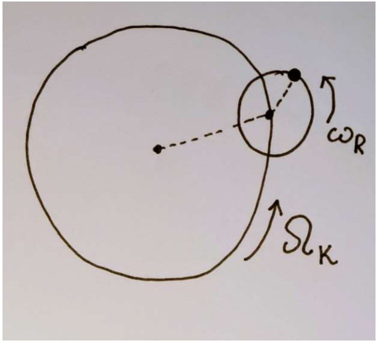
$$\left(\frac{\partial E_{\text{eff}}}{\partial r}\right)_\ell = 0$$

Epicyclic Frequency

$$\left(\frac{\partial^2 E_{\rm eff}}{\partial r^2}\right)_\ell = \omega^2 = \text{epicyclic frequency}$$

$$0 = \omega_R^2\,\delta r + \ddot{\delta r}$$

$$\omega_R = \Omega_K \left(1 - \frac{3r_H}{r}\right)^{1/2}$$



$\omega_R^2 > 0$ stability $\Rightarrow \dfrac{d\ell}{dr} > 0$

$\omega_R \neq \Omega_K \;\rightarrow$ **perihelion of Mercury**

Marginally Bound Orbit

$$\mathcal{E} = (u\eta) = \text{energy}$$

$$\mathcal{E} = \eta^i A(\eta_i + \Omega\xi_i) = A(\eta\eta) \qquad \text{Schwarzschild}$$

$$\mathcal{E}_K(\infty) = 1$$

But $\mathcal{E}_K = 1$ also at $r_{mb} = 2r_H$

Acceleration in ZAMO Frame

$$a_K = \gamma(n^i + V\tau^i)\nabla_i\{\gamma(n_K + V\tau_K)\} =$$

$$= \underbrace{V^0(\dots)_K}_{\text{gravity}} + \underbrace{V^1(\dots)_K}_{\text{Coriolis}} + \underbrace{V^2(\dots)_K}_{\text{centrifugal}}$$

$$= \nabla_K\Phi + \gamma^2 V\tilde{R}\nabla_K\omega + \gamma^2 V^2\tilde{R}^2\tilde{R}^{-1}\nabla_K\tilde{R}$$

In Schwarzschild

$$= \nabla_K\Phi + \gamma^2 V^2\tilde{r}^{-1}\nabla_K\tilde{r} \qquad \tilde{r} = \text{gyration radius}$$

The Photon Orbit

Schwarzschild: $\quad a_r = \partial_r \Phi + \gamma V^2 \frac{1}{\tilde{r}} \partial_r \tilde{r}$

$$\tilde{r} = \text{gyration radius} = r \left(1 - \frac{r_H}{r}\right)^{-1/2}$$

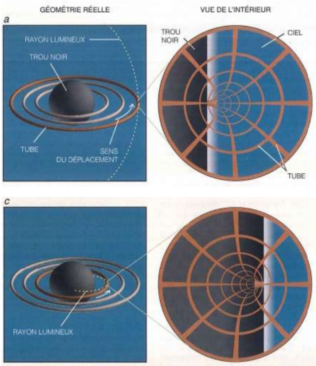
$$\frac{d\tilde{r}}{dr} = 0 \quad \text{at} \quad r_{ph} = \tfrac{3}{2} r_H \qquad \underline{\text{PHOTON ORBIT}}$$

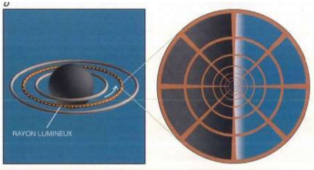
$r = \frac{3}{2} r_H$ circle is a geodesic in the optical 3-space,

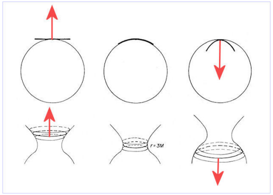
$$ds^2 = e^{\Phi} \left[dt^2 + dr_*^2 + \tilde{r}^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)\right]$$

r_* = tortoise coordinate, $\tilde{r}$ = gyration radius, Φ = gravity

The Optical Geometry







Several equations (Maxwell, Teukolsky) are <u>much</u> simpler in the optical geometry

$r < r_{ph}$: centrifugal and gravity forces have the same direction

Summary

		Schwarzschild
ISCO:	$\omega_R^2 = 0,\ \dfrac{d\ell}{dr} = 0$	$3r_H$
Marginally bound:	$\mathcal{E} = 1$	$2r_H$
Photon orbit:	$\dfrac{d\tilde{r}}{dr} = 0$	$\frac{3}{2}r_H$
Horizon		$r_H = \dfrac{2GM}{c^2}$

Paczyński–Wiita Potential

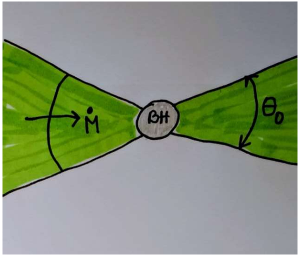
Newtonian hydrodynamics

$$\boxed{\Phi = -\frac{GM}{r - r_H}}$$

ISCO:	$\omega_R^2 = 0,\ \frac{d\ell}{dr} = 0$	$3r_H$
Marginally bound:	$\mathcal{E} = 0$	$2r_H$
Photon orbit		—
Horizon		$r_H = \frac{2GM}{c^2}$

END OF PART I

The Minimalistic Model: Żurek



CONICAL FLOW
No equation describing the vertical structure

$$u^\theta = 0,\ \partial_\theta = 0$$

$$T^i{}_k = \text{fluid} + \text{stress}, \qquad \text{stress} \ll \text{fluid}$$

$$T^i{}_k = \rho u^i u_k + \tau^i{}_k, \qquad \nabla_i(\rho u^i) = 0$$

Angular Momentum Equation

$$0 = \nabla_i T^i{}_k \xi^k = \frac{1}{\sqrt{-g}} \partial_i \left[\sqrt{-g}\, \rho u^i (u_\xi) + \tau^i{}_\varphi \right]$$

$$\sqrt{-g} = r^2, \qquad \text{conical flow:}$$

$$\partial_r \left[\underbrace{r^2 u^r \rho}_{\dot{M}/\theta_0} \underbrace{(u_\xi)}_{\mathcal{L}} + r^2 \underbrace{\tau^r{}_\varphi}_{\tau = \text{torque}} \right] = 0$$

$$-\frac{\dot{M}}{\theta_0} \mathcal{L} + r^2 \tau = -\frac{\dot{M}}{\theta_0} \mathcal{L}_H + r_H^2 \tau_H = 0$$

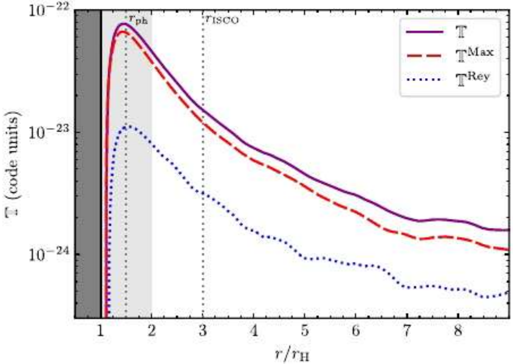
In Newton theory $\mathcal{L} = \ell$

The Stress Prescription

$$\frac{\dot{M}}{\theta_0}(\mathcal{L} - \mathcal{L}_H) = r^2 \tau$$

$$\boxed{\mathcal{L} = \frac{\theta_0}{\dot{M}} r^2 \tau + \mathcal{L}_H}$$

GRMHD simulations give $\tau(r)$ $\rightarrow$



Żurek: Conical Flow

$$\text{Polytrope } P = K\rho^{1+1/n}$$

$$a^2 = \frac{dP}{d\rho} = K(1+1/n)\rho^{1/n}$$

$$\rho = [K(1+1/n)]^{-n}\, a^{2n} = \mathcal{K} a^{2n}$$

$$\frac{1}{\rho}\frac{dP}{dr} = 2na\frac{da}{dr}$$

$$\text{Paczyński–Wiita} \qquad \Phi = -\frac{GM}{r - r_H}$$

Accretion Rate & Radial Force

Accretion rate:

$$\dot{M} = \theta_0 V \rho r^2 = \mathcal{K} r^2 V a^{2n}$$

$$\Downarrow$$

$$\boxed{a\frac{dV}{dr} + 2nV\frac{da}{dr} = -\frac{2aV}{r}}$$

Radial force:

$$\underbrace{V\frac{dV}{dr}}_{\text{acceleration}} + \underbrace{2na\frac{da}{dr}}_{\text{pressure}} + \underbrace{\frac{d\Phi}{dr}}_{\text{gravity}} + \underbrace{\frac{\mathcal{L}}{r^3}}_{\text{centrifugal}} = 0$$

$$\Downarrow$$

$$\boxed{V\frac{dV}{dr} + 2na\frac{da}{dr} = \frac{\mathcal{L}^2 - \ell_K^2}{r^3}}$$

2 ODEs of the Problem

$$a\frac{dV}{dr} + 2nV\frac{da}{dr} = -\frac{2aV}{r}$$

$$V\frac{dV}{dr} + 2na\frac{da}{dr} = -\frac{\mathcal{L}^2 - \ell_K^2}{r^3}$$

$$\Downarrow$$

$$\frac{dV}{dr} = \frac{V}{V^2 - a^2}\left[\frac{\mathcal{L}^2 - \ell_K^2}{r^3} + \frac{2a^2}{r}\right]$$

$$\frac{da}{dr} = \frac{-a}{2n(V^2 - a^2)}\left[\frac{\mathcal{L}^2 - \ell_K^2}{r^3} + \frac{2V^2}{r}\right]$$

The Regularity Conditions

At the sonic point $r = r_s$ where $a(r_s) = V(r_s)$ it should be

$$\ell_K^2(r_s) - \mathcal{L}^2(r_s) = 2a^2(r_s)\, r_s^2 = 2V^2(r_s)\, r_s^2$$

3 unknown constants: $\mathcal{L}_H,\ r_s,\ a(r_s) = V(r_s)$

and there are 3 equations at $r = r_s$ to fix them: mass accretion, integrated radial force, and regularity condition

The Żurek Problem Is Therefore Complete

Conclusions:

1. An analytic accretion disc model is fixed by $\dot{M}$ and the stress prescription (M and α in Shakura–Sunyaev language).
2. In the stationary case, regularity conditions at the sonic point are necessary to find the solution.