Black Holes as Non-Abelian Anyon Condensates

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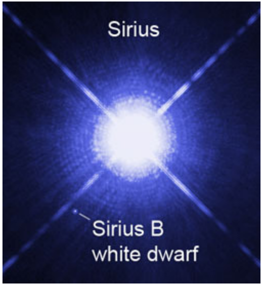
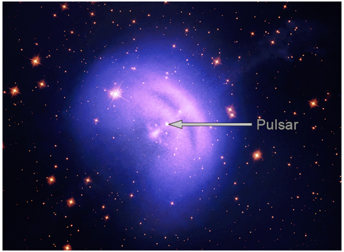
Motivation and gravitational framework

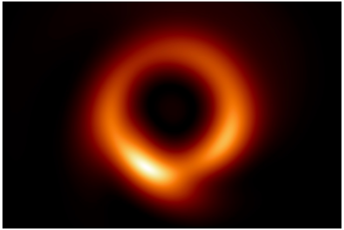
Typical stellar end states



White dwarf



Neutron star



Black hole

Two perspectives

How I Learned to Stop Worrying and Love the Horizon

- Collapse forms a trapped region
- Event horizon becomes a fundamental causal boundary
- Interior degrees of freedom become inaccessible to exterior observers
- Information paradox and 50 years of attempts to understand horizon physics

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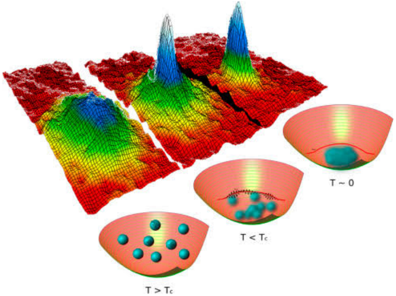
Paths to Regularity

- Collapse can stop before a trapped surface forms
- No fundamental event horizon
- No curvature singularity
- A new physical state replaces the interior

What would black holes be from a condensed matter perspective?



Bose–Einstein condensate

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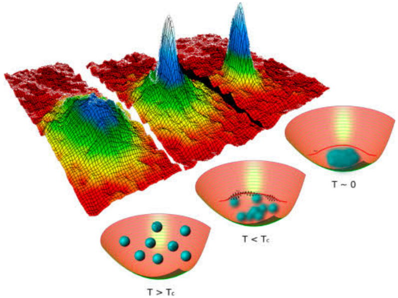

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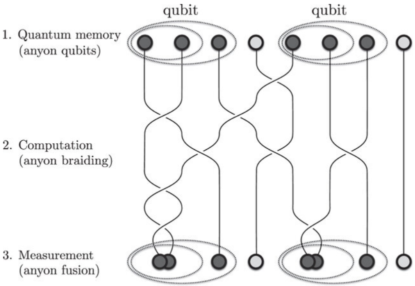

Anyons (2D due to spin-statistics)

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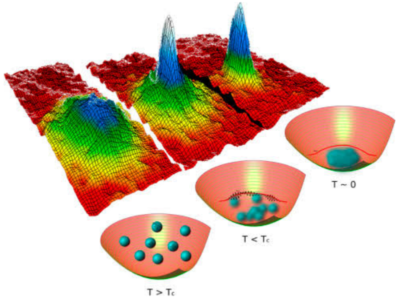

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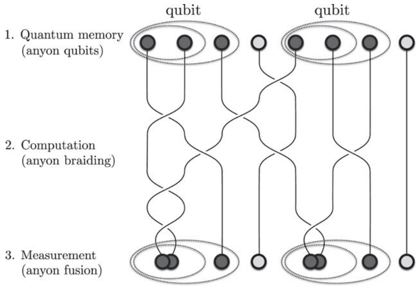

Anyons (2D due to spin-statistics)

Bekenstein first argued that black-hole entropy should scale with the horizon area,

$$S_{\mathrm{BH}} \propto A.$$

Hawking's calculation then fixed the coefficient, $S_{\mathrm{BH}} = \frac{A}{4}$.

Can gravity support a stable timelike shell?

Question: Can a gravitational theory support a stable, timelike material shell near the Schwarzschild radius?

General relativity: No, the Israel-shell stability requirement becomes singular as $r_c \to r_s$.

Beyond GR?

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Beyond GR? A generic higher-curvature theory may contain

$$\mathcal{L}_{\rm grav} = R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \cdots .$$

Which theory?
We need a principle that restricts this large space of possible gravitational actions.

Classical conformal symmetry

Following the standard Weyl/Mannheim convention,

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x), \qquad ds^2 \to \Omega^2 ds^2.$$

Matter fields transform by their conformal weights:

$$\phi \to \Omega^{-1}\phi, \qquad \psi \to \Omega^{-3/2}\psi, \qquad A_\mu \to A_\mu.$$

So in four dimensions, most Standard Model structures are already classically scale/conformally invariant:

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad i\bar{\psi}\gamma^\mu D_\mu\psi, \qquad (D_\mu H)^\dagger D^\mu H,$$

$$y\,\bar{\psi}_L H \psi_R, \qquad \lambda_H (H^\dagger H)^2, \qquad f^{abc} A^a A^b \partial A^c, \qquad g^2 A^4.$$

All these terms contain only dimensionless couplings:

$$g_i, \qquad y_f, \qquad \lambda_H.$$

Thus the massless gauge, fermion, Yukawa, and quartic scalar sectors are naturally compatible with a classical conformal starting point.

The exceptions

The explicit scale-breaking terms are

$$\frac{M_P^2}{2} R, \qquad \Lambda, \qquad -\mu_H^2 H^\dagger H$$

The Ricci scalar transforms in four dimensions as

$$R \to \Omega^{-2} \left[R - 6 \Box \ln \Omega - 6 \left(\nabla \ln \Omega \right)^2 \right]$$

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The Weyl tensor $C_{\mu\nu\rho\sigma}$ is the traceless part of the Riemann tensor. It transforms as

$$C^{\mu}{}_{\nu\rho\sigma} \to C^{\mu}{}_{\nu\rho\sigma}, \qquad C^2 \equiv C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \to \Omega^{-4}C^2.$$

Hence in four dimensions

$$\sqrt{-g}\,C^2 \to \sqrt{-g}\,C^2$$

so the Weyl-squared action is locally conformally invariant.

Scale-free origin of Einstein gravity

The Einstein term can be embedded in a locally conformal theory by introducing a scalar ϕ:

$$S_{\mathrm{conf}} = \int d^4x\sqrt{-g}\left[-\alpha_g C^2 + \frac{\xi}{2}\phi^2 R + \frac{1}{2}(\nabla\phi)^2 - \frac{\lambda}{4}\phi^4\right].$$

Gauge fixing $\phi = \mathrm{const.}$ gives the theory considered here,

$$\boxed{S_{\mathrm{grav}} = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \alpha_g C^2\right].}$$

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IF a nonzero scale could be generated,

$$\langle\phi\rangle = v_\phi,$$

then

$$\boxed{C^2 + \phi^2 R \;\longrightarrow\; C^2 + M_P^2 R + \cdots\,, \qquad M_P^2 = \xi v_\phi^2.}$$

This mirrors a superconductor:

$$F^2 + |D\Phi|^2 \to F^2 + m_A^2 A^2 + \cdots\,, \qquad m_A^2 = q^2 v^2.$$

In both cases, the condensate converts an already present field-dependent coupling into a constant physical scale.

Why Einstein–Weyl gravity?

For a local, parity-even, polynomial theory in four dimensions, with no fundamental scale and at most four derivatives,

$$\boxed{\text{local conformal invariance} \quad \Longrightarrow \quad C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}}$$

as the non-topological pure-gravity invariant. This parallels how Lorentz and gauge invariance select $F_{\mu\nu}F^{\mu\nu}$ in electromagnetism.
After a scale is generated, the gravitational response is schematically

$$h\left(M_P^2 k^2 + \alpha_g k^4\right) h, \qquad M_2^2 = \frac{M_P^2}{4\alpha_g}.$$

Hence

$$\boxed{k \ll M_2 : \ \text{Einstein regime}, \qquad k \gg M_2 : \ \text{Weyl regime.}}$$

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Quantum caveat. The fourth-order theory contains an additional spin-2 pole, conventionally associated with a ghost. Here C^2 is used only as an effective classical high-curvature response.

Next: can this response support a stable timelike shell near the Schwarzschild radius?

Timelike shells in general relativity

Young–Laplace equation: elementary form

For a spherical membrane of radius r:

$$F_{\mathrm{pressure}} = (p_{\mathrm{in}} - p_{\mathrm{out}})\,\pi r^2, \quad F_{\mathrm{tension}} = p_s(2\pi r).$$

Mechanical equilibrium,

$$F_{\mathrm{pressure}} = F_{\mathrm{tension}},$$

gives

$$(p_{\mathrm{in}} - p_{\mathrm{out}})\pi r^2 = 2\pi r\, p_s,$$

hence

$$\boxed{p_{\mathrm{in}} - p_{\mathrm{out}} = \frac{2p_s}{r}}.$$

For a general curved surface, with R_1, R_2 principle curvatures: $\frac{2}{r} \longrightarrow \frac{1}{R_1} + \frac{1}{R_2} = 2H$, so

$$\boxed{p_{\mathrm{in}} - p_{\mathrm{out}} = 2p_s H}.$$

Young–Laplace equation: relativistic form

Generalize the ordinary pressure to the normal stress:

$$p_{r,\mathrm{in}} \equiv T^{\mathrm{in}}_{\mu\nu} n^\mu n^\nu, \qquad p_{r,\mathrm{out}} \equiv T^{\mathrm{out}}_{\mu\nu} n^\mu n^\nu,$$

where $T_{\mu\nu}$ is the bulk stress–energy tensor.
For a perfect fluid at rest relative to the membrane,

$$p_{r,\mathrm{in}} = p_{\mathrm{in}}, \qquad p_{r,\mathrm{out}} = p_{\mathrm{out}}.$$

The membrane carries a surface stress–energy tensor

$$\boxed{S_{\hat{a}\hat{b}} = \mathrm{diag}(\sigma, -p_s, -p_s)},$$

where σ is the surface energy density and $p_s > 0$ is the tangential surface tension. The angular curvature gives the Young–Laplace force $2p_s H$, while the temporal extrinsic curvature gives the inertial contribution

$$\sigma K^\tau{}_\tau.$$

Therefore,

$$\boxed{p_{r,\mathrm{in}} - p_{r,\mathrm{out}} = \sigma K^\tau{}_\tau + 2p_s H}.$$

Shell stability – GR geometry

Take a spherical shell $r = r(\tau)$ separating two static spherical geometries,

$$ds_i^2 = B_i(r)\, dt_i^2 - \frac{dr^2}{B_i(r)} - r^2 d\Omega^2, \qquad i = \mathrm{in}, \mathrm{out}.$$

For the GR comparison,

$$\boxed{B_{\mathrm{in}}(r) = 1, \qquad B_{\mathrm{out}}(r) = f(r) = 1 - \frac{r_s}{r}}$$

corresponding to Minkowski inside and Schwarzschild outside.
Both bulk regions are vacuum:

$$\boxed{p_{r,\mathrm{in}} = p_{r,\mathrm{out}} = 0.}$$

For a spherical shell,

$$K^\tau{}_{\tau,i} = \frac{\ddot{r} + \frac{1}{2} B_i'(r)}{\sqrt{\dot{r}^{\,2} + B_i(r)}}, \qquad H_i = \frac{\sqrt{\dot{r}^{\,2} + B_i(r)}}{r}.$$

The Young–Laplace equation uses the averages

$$K^\tau{}_\tau = \frac{1}{2}\left(K^\tau{}_{\tau,\mathrm{in}} + K^\tau{}_{\tau,\mathrm{out}}\right), \qquad H = \frac{1}{2}\left(H_{\mathrm{in}} + H_{\mathrm{out}}\right).$$

Shell stability – Young–Laplace balance

The GR shell has

$$S_{\hat{a}\hat{b}} = \mathrm{diag}(\sigma_{\mathrm{GR}}, -p_s, -p_s),$$

so the generalized Young–Laplace equation is

$$p_{r,\mathrm{in}} - p_{r,\mathrm{out}} = \sigma_{\mathrm{GR}} K^{\tau}{}_{\tau} + 2 p_s H.$$

Since $p_{r,\mathrm{in}} = p_{r,\mathrm{out}} = 0$,

$$0 = \frac{\sigma_{\mathrm{GR}}}{2}\left[\frac{\ddot{r}}{\sqrt{\dot{r}^{\,2}+1}} + \frac{\ddot{r} + \frac{1}{2} f'(r)}{\sqrt{\dot{r}^{\,2}+f(r)}}\right] + \frac{p_s}{r}\left[\sqrt{\dot{r}^{\,2}+1} + \sqrt{\dot{r}^{\,2}+f(r)}\right].$$

The static background is

$$r = r_c, \qquad \dot{r} = \ddot{r} = 0, \qquad \delta \equiv f(r_c) = 1 - \frac{r_s}{r_c}.$$

The Young–Laplace equation evaluated at r_c fixes the equilibrium relation between σ_{GR} and p_s.

Shell stability – radial perturbation

Perturb the equilibrium radius,

$$r(\tau) = r_c + \xi(\tau).$$

For the vacuum Israel shell,

$$p_{r,\mathrm{in}} = p_{r,\mathrm{out}} = 0 \quad \Longrightarrow \quad p_s < 0.$$

Hence it is pressure-supported, with

$$p_t \equiv -p_s > 0, \qquad \eta_{\mathrm{GR}} \equiv \left.\frac{dp_t}{d\sigma_{\mathrm{GR}}}\right|_{r_c}.$$

Linearization gives

$$\ddot{\xi} + \omega_{\mathrm{GR}}^2 \xi = 0, \qquad \omega_{\mathrm{GR}}^2 > 0 \Longrightarrow \boxed{\eta_{\mathrm{GR}} > \eta_{\mathrm{GR}}^{\mathrm{crit}}}.$$

$$\eta_{\mathrm{GR}}^{\mathrm{crit}}(\delta) = \frac{1 + \sqrt{\delta} + \delta - 3\delta^{3/2}}{4\delta(1 + 3\sqrt{\delta})}.$$

Near the Schwarzschild radius,

$$\boxed{\eta_{\mathrm{GR}}^{\mathrm{crit}} \sim \frac{1}{4\delta} \to +\infty \qquad (\delta \to 0^+).}$$

Einstein–Weyl membrane stability

Shell stability – Einstein–Weyl background

Start from

$$M_P^2 G^\mu{}_\nu + 4\alpha_g W^\mu{}_\nu = T^\mu{}_\nu, \qquad W^\mu{}_\mu = 0.$$

For

$$ds^2 = B(r)dt^2 - \frac{dr^2}{B(r)} - r^2 d\Omega^2,$$

the field equation implies

$$\boxed{(r^4 B^{(3)})' = \frac{3r^4}{4\alpha_g B}(\rho + p_r)}$$

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For a vacuum-like interior,

$$T^\mu{}_\nu\big|_{\rm in} = \rho_{\rm in}\,\delta^\mu{}_\nu, \qquad p_{r,\rm in} = p_{t,\rm in} =: p_{\rm in} = -\rho_{\rm in},$$

so

$$(r^4 B_{\rm in}^{(3)})' = 0 \quad \Longrightarrow \quad B_{\rm in} = \frac{b_{-1}}{r} + b_0 + b_1 r + b_2 r^2.$$

Regularity gives $b_{-1} = b_1 = 0$ and $b_0 = 1$. Define the geometric interior curvature scale by

$$\boxed{B_{\rm in}(r) = 1 - \frac{\Lambda_{\rm in}}{3} r^2, \qquad R_{\rm in} = 4\Lambda_{\rm in}}$$

with

$$\rho_{\rm in} = -M_P^2 \Lambda_{\rm in}, \qquad p_{\rm in} = M_P^2 \Lambda_{\rm in}.$$

Shell stability – conformal membrane

Take the Schwarzschild exterior

$$f(r) \equiv B_{\rm out}(r) = 1 - \frac{r_s}{r}, \qquad \boxed{B_c \equiv B_{\rm out}(r_c) = \delta = 1 - \frac{r_s}{r_c}}$$

Define the projected proper source

$$\Sigma \equiv \int (\rho + p_r)\, d\ell, \qquad \Sigma(r) = \frac{4\alpha_g}{3} \sqrt{B_{\rm out}(r)}\, B^{(3)}_{\rm out}(r)$$

The physical surface variables are

$$\sigma_{\rm CG} \equiv \int \rho\, d\ell, \qquad p_s \equiv - \int p_t\, d\ell > 0$$

Parametrize the radial stress by

$$\int p_r\, d\ell = -a\, p_s,$$

so that

$$\boxed{\sigma_{\rm CG} = \Sigma + a\, p_s}, \qquad S_{\hat a \hat b} = \mathrm{diag}(\sigma_{\rm CG}, -p_s, -p_s)$$

Shell stability – Young–Laplace balance

The normal-force balance is

$$p_{r,\mathrm{in}} - p_{r,\mathrm{out}} = \sigma_{\mathrm{CG}} K^{\tau}{}_{\tau} + 2 p_s H.$$

Using

$$B_{\mathrm{in}}(r_c) \simeq 1, \qquad B_{\mathrm{out}} = f(r) = 1 - \frac{r_s}{r},$$

gives

$$p_{r,\mathrm{in}} - p_{r,\mathrm{out}} = \frac{\sigma_{\mathrm{CG}}}{2} \left[\frac{\ddot{r}}{\sqrt{\dot{r}^2 + 1}} + \frac{\ddot{r} + f'/2}{\sqrt{\dot{r}^2 + f}} \right] + \frac{p_s}{r} \left[\sqrt{\dot{r}^2 + 1} + \sqrt{\dot{r}^2 + f} \right].$$

At equilibrium,

$$r = r_c, \qquad \dot{r} = \ddot{r} = 0, \qquad \delta = f(r_c),$$

and for the leading radial mode,

$$\frac{d}{dr} (p_{r,\mathrm{in}} - p_{r,\mathrm{out}}) \Big|_{r_c} = 0.$$

The nonzero normal support permits genuine tangential tension $p_s > 0$, unlike the pressure-supported vacuum Israel shell.

Shell stability – radial perturbation

Perturb $r(\tau) = r_c + \xi(\tau)$. Since the conformal membrane is tension-supported, define

$$\eta_{\rm CG} \equiv \left.\frac{dp_s}{d\sigma_{\rm CG}}\right|_{r_c}.$$

From $\sigma_{\rm CG} = \Sigma + ap_s$,

$$\frac{d\sigma_{\rm CG}}{dr} = \frac{\Sigma'}{1 - a\eta_{\rm CG}}, \qquad \frac{dp_s}{dr} = \frac{\eta_{\rm CG}\Sigma'}{1 - a\eta_{\rm CG}},$$

with

$$\Sigma_c' = \frac{\Sigma_c}{2r_c\delta}(1 - 9\delta).$$

Linear stability,

$$\ddot{\xi} + \omega_{\rm CG}^2\xi = 0, \qquad \omega_{\rm CG}^2 > 0,$$

gives near the exterior root

$$\boxed{\frac{1}{a} - \frac{\Sigma_c}{a^2 p_s} + O(\delta) < \eta_{\rm CG} < \frac{1}{a}}.$$

For isotropy, $a = 1$,

$$\boxed{1 - \frac{\Sigma_c}{p_s} + O(\delta) < \eta_{\rm CG} < 1, \qquad \eta_{\rm CG}^{\rm crit} \to 1}.$$

Shell stability – comparison

Buchdahl bound:

$$R > \frac{9}{8} r_s$$

for an ordinary static perfect-fluid star.

Israel thin shell:

$$\eta_{\mathrm{GR}}^{\mathrm{crit}} \sim \frac{1}{4\delta} \to +\infty \qquad (\delta \to 0^+).$$

A near-horizon vacuum shell requires an increasingly large pressure response.

Conformal membrane:

$$\frac{1}{a} - \frac{\Sigma_c}{a^2 p_s} + O(\delta) < \eta_{\mathrm{CG}} < \frac{1}{a}, \qquad \eta_{\mathrm{CG}} \equiv \frac{dp_s}{d\sigma_{\mathrm{CG}}}.$$

The near-horizon response is finite.

The conformal branch ends when

$$\delta = \frac{1}{9} \quad \Longleftrightarrow \quad r_c = \frac{9}{8} r_s.$$

Thus

$$\boxed{r_s < r_c < \frac{9}{8} r_s}$$

is the regular stable window: below the Buchdahl radius, but not including the endpoint.

Consequences and anyonic interpretation

Physical endpoint

- Collapse is assumed to terminate in a two-dimensional condensate of non-Abelian anyons.
- The microscopic degrees of freedom live on a timelike shell near the would-be horizon.

regular interior	anyon shell at r_c	Schwarzschild-like exterior

- The shell carries the black-hole entropy and replaces the singular interior as the microscopic system.
- The relevant many-body data are the fusion Hilbert space, the shell area, and the quantum dimension d.

$$\boxed{\dim \mathcal{H}_N \propto d^N}$$

Fusion entropy fixes the area quantum

- Fusion-state counting gives the microscopic entropy of N shell constituents.

$$S_{\mathrm{fus}}(N) = N \log d + O(1), \qquad A = NA_0.$$

$$S_{\mathrm{fus}}(A) = \frac{A}{A_0} \log d + O(1).$$

$$\boxed{A_0 = 4\ell_P^2 \log d}$$

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- The same area quantum fixes the mass spectrum of a Schwarzschild exterior.

$$A = 16\pi M^2, \qquad A_n = nA_0.$$

$$\boxed{M_n = \mu\sqrt{n}, \qquad \mu^2 = \frac{\log d}{4\pi}}$$

Transition scale and Hawking temperature

- Adjacent shell levels have energy spacing

$$\Delta E_n = M_n - M_{n-1} = \mu\left(\sqrt{n} - \sqrt{n-1}\right).$$

$$\Delta S = \log d, \qquad T_n = \frac{\Delta E_n}{\Delta S}.$$

$$T_n = \frac{\mu}{\log d}\left(\sqrt{n} - \sqrt{n-1}\right).$$

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$$T_n = \frac{\mu}{\log d}\left(\sqrt{n} - \sqrt{n-1}\right).$$

- Expanding at large n gives the Hawking scale with controlled corrections.

$$\boxed{T(M) = \frac{1}{8\pi M}\left[1 + \frac{\log d}{16\pi M^2} + \frac{(\log d)^2}{128\pi^2 M^4} + \cdots\right]}$$

Thermodynamic entropy

- Integrating $dS = dM/T(M)$ gives the entropy in terms of the ADM mass.

$$S_{\mathrm{therm}}(M) = 4\pi M^2 - \frac{\log d}{2} \log M + \frac{(\log d)^2}{64\pi M^2} + \cdots .$$

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$$A = 16\pi M^2 .$$

- In area variables this becomes the entropy target for the effective Hamiltonian.

$$\boxed{S_{\mathrm{therm}}(A) = \frac{A}{4} - \frac{\log d}{4} \log A + \frac{(\log d)^2}{4A} + \cdots}$$

Equipartition

- The local shell modes may be treated classically when their level spacing is small compared with the thermal scale, $\dfrac{\Delta E}{N} \ll k_B T$.

$$N_{\mathrm{dof}} := \frac{A}{\ell_P^2}, \qquad E_{\mathrm{therm}} = M.$$

$$E_{\mathrm{therm}} = \frac{1}{2} N_{\mathrm{dof}} T \quad \to \quad \boxed{T = \frac{2M}{A} = \frac{1}{8\pi M}}$$

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- For charged and rotating families the same structure is recovered with the appropriate thermal energy.

$$E^{\rm Kerr}_{\rm therm} = M\sqrt{1-a^2}, \qquad E^{\rm RN}_{\rm therm} = M\sqrt{1-q^2}.$$

$$\boxed{T = \frac{2E_{\rm therm}}{A}}$$

Collective Hamiltonian

- Let each shell constituent carry m collective components $x_i \in \mathbb{R}^m$.

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i, \qquad \tilde{r}^{\,2} = \frac{1}{N}\sum_{i=1}^{N} \|x_i - \bar{x}\|^2.$$

$$r_0^2 = \frac{\ell_P^2 \log d}{\pi}.$$

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$$r_0^2 = \frac{\ell_P^2 \log d}{\pi}.$$

- The minimal Hamiltonian fixes relative fluctuations and removes the center mode.

$$\boxed{H = \frac{\kappa}{2}\sum_{i=1}^{N} \|x_i - \bar{x}\|^2 + \frac{\lambda N}{4}\left(\tilde{r}^{\,2} - r_0^2\right)^2 + \frac{\eta N^{p+1}}{2}\|\bar{x}\|^2}$$

Canonical entropy from the Hamiltonian

- The large-N canonical entropy of the collective Hamiltonian is

$$\boxed{S_{\mathrm{can}}(N) = N\frac{m}{2}\left(1 + \log\frac{2\log d}{m}\right) - \frac{pm}{2}\log N + \cdots}$$

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$$\boxed{S_{\mathrm{can}}(N) = N\frac{m}{2}\left(1+\log\frac{2\log d}{m}\right) - \frac{pm}{2}\log N + \cdots}$$

- The thermodynamic entropy written in N is

$$\boxed{S_{\mathrm{therm}}(N) = N\log d - \frac{\log d}{4}\log N + \cdots}$$

Quantum dimension from entropy matching

- Matching the leading extensive terms fixes the semiclassical degeneracy scale.

$$N\frac{m}{2}\left(1+\log\frac{2\log d}{m}\right)=N\log d.$$

$$\boxed{\log d=\frac{m}{2},\qquad d=\left(\sqrt{e}\right)^{m}}$$

- Matching the logarithmic terms fixes the center-mode exponent.

$$\frac{pm}{2}=\frac{\log d}{4}=\frac{m}{8}.$$

$$\boxed{p=\frac{1}{4}}$$

Algebraic anyon dimensions

- The values $d = (\sqrt{e})^m$ are semiclassical targets, not exact microscopic quantum dimensions.
- Exact anyon quantum dimensions are algebraic.

$$\boxed{d_j^{(k)} = \frac{\sin[(2j+1)\pi/(k+2)]}{\sin[\pi/(k+2)]}}$$

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- Nearby algebraic values can realize the same effective scale.

m	target $e^{m/2}$	nearby algebraic candidate
1	1.65	Fibonacci: $d = \varphi \simeq 1.62$
2	2.72	$SU(2)_{10}$: $d = 1 + \sqrt{3} \simeq 2.73$
3	4.48	$SU(2)_{12}$: $d \simeq 4.49$
4	7.39	$SU(2)_{22}$: $d \simeq 7.40$

Discrete transition frequencies

- Absorption raises the shell from $n - k$ to n.

$$\omega_{n,k} = M_n - M_{n-k}.$$

$$\boxed{\omega_{n,k} = M - \sqrt{M^2 - \frac{k \log d}{4\pi}}}$$

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- For $M \gg 1$, the allowed transition lines are approximately evenly spaced.

$$\boxed{\omega_{n,k} \simeq kT_H \log d, \qquad \Delta\omega \simeq T_H \log d}$$

Absorption versus reverse emission

- The degeneracy ratio between the two shell levels is fixed by the fusion space.

$$g_n \propto d^n, \qquad \frac{g_{n-k}}{g_n} = d^{-k}.$$

$$\boxed{\frac{\Gamma^{\mathrm{em}}_{n\to n-k}}{\Gamma^{\mathrm{abs}}_{n-k\to n}} \simeq d^{-k}}$$

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- The same factor can be written as a Boltzmann weight at the transition frequency.

$$\exp\left(-\frac{\omega_{n,k}}{T_H}\right) \simeq \exp(-k \log d) = d^{-k}.$$

$$\boxed{e^{-\omega_{n,k}/T_H} \simeq d^{-k}}$$

Reflectivity and absorptivity

- With the fluctuation–dissipation identification, the transition weight becomes a line reflectivity.

$$\boxed{|\mathcal{R}_{n,k}|^2 \simeq d^{-k}, \qquad \mathcal{A}_{n,k} \simeq 1 - d^{-k}}$$

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- The same response gives an effective membrane viscosity.

$$\boxed{\frac{\eta_{\mathrm{visc}}^{\mathrm{eff}}(\omega_{n,k})}{\eta_{\mathrm{BH}}} \simeq \frac{1 - d^{-k/2}}{1 + d^{-k/2}} = \tanh\left(\frac{k \log d}{4}\right)}$$

Resolved lines versus thermal envelope

- The anyonic line structure is observable only when the linewidths are smaller than the spacing.

$$\boxed{\Gamma_{n,k} \ll \Delta\omega \quad \Longrightarrow \quad \text{resolved anyonic response}}$$

- If neighboring lines overlap, the response becomes a smooth Boltzmann envelope.

$$\boxed{|\mathcal{R}_{\rm sh}(\omega)|^2 \simeq e^{-\omega/T_H}, \qquad \mathcal{A}_{\rm sh}(\omega) \simeq 1 - e^{-\omega/T_H}}$$

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- An inward quantum is reabsorbed after only order-one shell encounters.

$$\boxed{N_{\rm hit}^{(k)} \simeq \frac{1}{1-d^{-k}} \leq \frac{d}{d-1}}$$

Echo delay and echo suppression

- The delay is set by the near-root tortoise coordinate and measures the shell location.

$$\boxed{\Delta t_{\mathrm{echo}} \simeq \kappa_s^{-1} \log \frac{1}{\delta}}$$

For Schwarzschild,

$$\boxed{\Delta t_{\mathrm{echo}} \simeq 8M \log\left(\frac{4M}{\ell_c}\right) + O(M)}$$

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- The echo amplitude is suppressed by both shell reflectivity and the exterior curvature barrier.

$$\boxed{E_j(\omega_{n,k}) \sim \left[d^{-k} |\mathcal{R}_{\mathrm{bar}}(\omega_{n,k})|^2\right]^j E_0}$$

Consequences

- The same microscopic parameter controls the line spacing and the line weights.

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$$\boxed{\Delta\omega \simeq T_H \log d, \qquad |\mathcal{R}_{n,k}|^2 \simeq d^{-k}}$$

- The echo delay probes a separate geometric parameter.

$$\boxed{\Delta t_{\mathrm{echo}} \to \ell_c, \qquad \Delta\omega,\ |\mathcal{R}_{n,k}|^2 \to d}$$

Illustrative observational proxy

Input: identify the dominant GW250114 ringdown mode with the lowest Kerr area transition, taking $m_{\rm az} = 2$ and a single area step $k = 1$,

$$\omega_{220}^{\rm obs} \simeq \omega_{2,1}^{\rm trans} = 2\Omega_H + \frac{\alpha_{\rm BM}\kappa_s}{8\pi} \;\Rightarrow\; \alpha_{\rm BM} \equiv \frac{A_0}{\ell_P^2} = 4\log d = 2m_{\rm col}.$$

Using the published GW250114 remnant parameters gives, approximately,

$$\alpha_{\rm BM} \simeq 16.5, \qquad m_{\rm eff} \simeq 8.25.$$

Nearest semiclassical branch:

$$\boxed{m = 8}, \qquad \boxed{d \simeq d_{81/2}^{(170)} = 54.6063}.$$

After fixing this value of d:

$$\boxed{|\mathcal{R}_1|^2 = \frac{1}{d} = 0.0183}, \qquad \boxed{\mathcal{A}_1 = 1 - \frac{1}{d} = 0.9817}.$$

Signal-size comparison

Using the GW250114 postmerger value

$$\rho_{\mathrm{post}} \simeq 26,$$

the optimistic signal scale is

$$\boxed{\rho_{\mathrm{echo}}^{\mathrm{max}} \lesssim \frac{26}{\sqrt{54.6063}} \simeq 3.5.}$$

Current model-agnostic echo searches report

$$\boxed{\rho_{\mathrm{echo}} < 6.38 \quad (90\%)}.$$

Thus

$$\boxed{3.5 < 6.38}.$$

Present data do not exclude this signal scale.


Thank you!