The Friedman model

The metric

$$ds^2 = c^2dt^2 - R^2(t)(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2))$$

The Einstein field equations

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{8\pi G}{c^2}p = -\frac{kc^2}{R^2} + \Lambda c^2\,, \tag{1}$$

$$\frac{\dot{R}^2}{R^2} - \frac{8\pi G}{3}\varrho = -\frac{kc^2}{R^2} + \frac{1}{3}\Lambda c^2\,, \tag{2}$$

subtracting the second equation from the first, we get

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\varrho + \frac{3p}{c^2}) + \frac{\Lambda c^2}{3}\,, \tag{3}$$

some time ago we introduced the Hubble constant via $v = H \cdot d$,

it turns out that it is connected with the rate of change of the scale factor R(t)

$$H(t) = \frac{\dot{R}(t)}{R(t)}\,. \tag{4}$$

Let us introduce a so called deceleration parameter q(t), defined as

$$q(t) = -\frac{\ddot{R}R}{\dot{R}^2}\,, \tag{5}$$

equation (3) can be rewritten as

$$(\varrho + \frac{3p}{c^2}) - \frac{\Lambda c^2}{4\pi G} = \frac{3H^2q}{4\pi G}\,, \tag{6}$$

equation (2) can be rewritten as

$$\frac{kc^2}{R^2} = \frac{1}{3}(8\pi G\varrho + \Lambda c^2) - H^2\,, \tag{7}$$

Using equation (6) can be transformed into:

$$\frac{kc^2}{R^2} = \frac{4\pi G}{3q}(\varrho(2q-1) - \frac{3p}{c^2}) + \frac{\Lambda c^2}{3}(1 + \frac{1}{q})\,. \tag{8}$$

H and q are the basic parameters characterizing dynamics of the universe.

How to determine H and q ?

Let us recall the relation $1 + z = \frac{R(t_0)}{R(t_e)}$

$$1 + z = \frac{R(t_0)}{R(t_0 - \Delta t)} = 1 + \Delta\frac{\dot{R}(t_0)}{R(t_0)} + \Delta^2(\frac{\dot{R}_0^2}{{R_0}^2} - \frac{\ddot{R}_0}{2R_0})\,,$$

flux $l = \dfrac{L}{4\pi r^2 {R^2}_0(1+z)^2}$ so the luminosity distance $d_L = (\dfrac{L}{4\pi l})^{1/2} = rR_0(1+z)$ ,

for small z, $c \cdot z = H \cdot d_L$

In astronomy the distance-magnitude relation is usually used $m = 5\log D + M - 5$.

Using the luminosity distance this relation can be transformed into (not easy!)

$$m - M = 5\log\frac{cz}{H_0} + 1.086(1-q_0)z - 0.27(1-q_0)(1+3q_0)z^2 + 25$$

Some exact solutions of the Friedman equations

Let us consider pressureless gas in a flat universe with $\Lambda = 0$

In this case the equation (2) reduces to

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\varrho\,. \tag{9}$$

The energy-momentum conservation law ${T^{ab}}_b = 0$, reduces to:

$$\varrho \cdot R^3 = \text{const}\,. \tag{10}$$

From equations (9) and (10), it follows that

$$R(t) \propto t^{2/3}\,.$$

In such matter dominated universe $H(t) = \dfrac{2}{3t}$ and $q = \frac{1}{2}$.

Let us discuss the first obvious consequences:

$R(t) \sim t^{2/3}$ implies that when $t \to 0$, $R(t) \to 0$

so, the Universe had a beginning !

Since $\varrho \cdot R^3$=const, when $R \to 0$, $\varrho \to \infty$ !!!

Early in 1940-ties George Gamow realized that if the early Universe was very dense it was also very hot. So let us consider radiation dominated Universe.

Basic thermodynamical properties of radiation:

$$\varepsilon_{rad} = a \cdot T^4, \ a - \text{Stefan} - \text{Boltzmann constant}, \ p_{rad} = \frac{1}{3}\varepsilon_{rad} \, . \tag{11}$$

From the energy-momentum conservation law it follows that:

$$\frac{d}{dt}(\varepsilon_{rad} R^3) + p\frac{d}{dt}(R^3) = 0, \text{ so}$$

$$\frac{d}{dt}(\varepsilon_{rad} R^3) + \frac{1}{3}\varepsilon_{rad}\frac{d}{dt}(R^3) = 0 \to$$

$$\frac{d}{dt}(\varepsilon_{rad} \cdot R^4) = 0, \ \to \varepsilon_{rad} \cdot R^4 = \text{const, or } T \cdot R = \text{const} \, .$$

Equation (2) assumes now the form:

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\varepsilon_{rad} \, , \text{or } \frac{\dot{R}^2}{R^2} =\sim \frac{1}{R^4} \, ,$$

what leads to:

$$R(t) \sim t^{1/2} \, .$$

It means that when $R(t) \to 0, \quad T(t) \to \infty$ !!!

The early Universe was very dense and very hot !!!

In such radiation dominated universe $H(t) = \dfrac{1}{2t}$ and $q = 1$.

Finally let us consider the flat, empty Universe with $\Lambda \neq 0$ .

In this case equation (2) assumes a simple form:

$$\frac{\dot{R}^2}{R^2} = \frac{1}{3}\Lambda c^2 \, ,$$

that leads to an exponential solution:

$$R(t) \sim \exp(\sqrt{\frac{\Lambda c^2}{3}} \cdot t)\,.$$

In such dark energy dominated universe $H(t) = \sqrt{\frac{\Lambda c^2}{3}}$ and $q = 1$.

So summarizing we have:

$$R(t) \sim \begin{cases} t^{2/3}\,, & \text{matter dominated universe}\,, \\ t^{1/2}\,, & \text{radiation dominated universe}\,, \\ \exp(\sqrt{\frac{\Lambda c^2}{3}} \cdot t)\,, & \text{dark energy dominated universe}\,. \end{cases}$$

The notion of critical density allows convenient parametrization of the Hubble constant:

$$H(z) = H_0\sqrt{\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda}\,, \text{ where}$$

$\Omega_r$ - represents contribution of radiation, $\Omega_m$ - matter,
$\Omega_k$ - curvature, and $\Omega_\Lambda$ - cosmological constant or Dark Energy.

This relation implies that if $\Omega_r \neq 0$ the early evolution of the Universe was dominated by radiation.

Since $H(z=0) = H_0$ we also have a constrain:

$$\Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda = 1$$

Current values:

$$\Omega_r = 2.47 \cdot 10^{-5} h^{-2},\ \Omega_m = 0.315 \pm 0.007,\ \Omega_\Lambda = 0.685 \pm 0.007\,.$$

Using the Friedman equations and the Hubble law it is possible to calculate how much time a light signal emitted at z needed to reach us

$$1 + z = \frac{R(t_0)}{R(t_e)},\ \text{let us use a common convention } R(t_0) = 1,\ R(t_e) = R(t)\,,$$

$$\frac{\dot{R}^2}{R^2} - \frac{8\pi G}{3}\varrho = -\frac{kc^2}{R^2} + \frac{1}{3}\Lambda c^2\,.$$

This equation can be rewritten as:

$$H(z) = H_0\sqrt{\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda}\,.$$

$$\text{Using } 1+z = \frac{1}{R(t)}\,, \text{ and } H = \frac{\dot{R}}{R}\,, \text{ we find}$$

$$t(z) = \frac{1}{H_0}\int_0^z \frac{dz}{(1+z)\sqrt{\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda}}\,.$$

Energy scale:
Energy: 1 Gev = $1.6022{\cdot}10^{-10}$ J
Temperature: 1 GeV = $1.605{\cdot}10^{13}$ K
Proton mass: 938.272 MeV
Neutron mass: 939.566 MeV
Electron mass: 0.5110 Mev

Composition of matter at $T \sim 0.2$ GeV

Photons, neutrinos, electrons and positrons, miuons, and tauons are in thermal equilibrium

$$\gamma + \gamma \leftrightarrow \mu^+ + \mu^- \leftrightarrow \nu_\mu + \tilde{\nu}_\mu \leftrightarrow \gamma + \gamma$$

$$\gamma + \gamma \leftrightarrow \tau^+ + \tau - \leftrightarrow \nu_\tau + \tilde{\nu}_\tau \leftrightarrow \gamma + \gamma$$

$$\gamma + \gamma \leftrightarrow e^+ + e^- \leftrightarrow \nu_e + \tilde{\nu}_e \leftrightarrow \gamma + \gamma$$

Reactions between leptons, like

$$e^+ + \mu^- \leftrightarrow \tilde{\nu}_e + \nu_\mu\,,\ e^- + \mu^+ \leftrightarrow \nu_e + \tilde{\nu}_\mu\,,$$

$$e^+ + \nu_e \leftrightarrow \mu^+ + \nu_\mu\,,\ e^- + \tilde{\nu}_e \leftrightarrow \mu_- + \tilde{\nu}_\mu\,.$$

preserve the state of thermal equilibrium.

Reaction rates between different leptons are determined by:

$$\Gamma_{e\nu} = n_e c \sigma_{e\nu} \quad \Gamma_{\mu\nu} = n_\mu c \sigma_{\mu\nu}\,, \text{where}$$

$\sigma_{e\nu}$ and $\sigma_{\mu\nu}$ denote the cross sections for the appropriate reactions.
The state of thermal equilibrium is maintained when $\Gamma(T)\cdot t(T) \gg 1$.
When this condition is violated, reactions with neutrinos
are too slow to maintain thermal equilibrium.
At the moment of freeze out, the number density of neutrinos is $n_\nu = \frac{3}{8} g_\nu n_\gamma$,
where $g_\nu = \Sigma g_{\nu i}$ denotes the total number of spin states.
At that epoch neutrons and protons are still in a state of thermal equilibrium,
that is kept due to the reactions:

$$e_+ + n \leftrightarrow p + \tilde{\nu}_e\,,$$

$$\nu_e + n \leftrightarrow p + e^-\,,$$

$$n \leftrightarrow p + e^- + \tilde{\nu}_e \,.$$

However at certain temperature $T_* \approx 1MeV$, $\Gamma(T_*) \cdot t(T_*) \approx 1$ and
the number density of neutrons relative to protons becomes frozen at a level of:

$$\frac{n_n}{n_p} \approx \left(\frac{n_n}{n_p}\right)_* = \exp(-\frac{\Delta mc^2}{k_B T_*}) \approx 0.27 \,.$$

However free neutrons are unstable and they start to decay,
so at onset of the primordial nucleosynthesis $\frac{n_n}{n_p} \approx 0.14$

The state of thermal equilibrium

All particles, depending on their spins, are described by the Fermi-Dirac or Bose-Einstein phase space distribution:

$$f(\vec{p}) = [\exp((E-\mu)/T \pm 1]^{-1} \,,$$

where $E$ - denotes energy, $\mu$ chemical potential, and the Boltzmann constant was set to be equal $k_B = 1$. The number density $n$, energy density $\varrho$ and pressure $p$ of a dilute, weakly interacting gas of particles with $g$ internal degrees of freedom is given by;

$$n = \frac{g}{(2\pi)^3} \int f(\vec{p}) d^3p \,,$$

$$\varrho = \frac{g}{(2\pi)^3} \int E(\vec{p}) f(\vec{p}) d^3p \,,$$

$$p = \frac{g}{(2\pi)^3} \int \frac{|\vec{p}|^2}{3E} f(\vec{p}) d^3p \,,$$

where $E^2 = |\vec{p}|^2 + m^2$.

In kinetic equilibrium, the number density of a nonrelativistic nuclear species $A(Z)$ with mass number $A$ and charge Z is given by

$$n_A = g_A (\frac{m_A T}{2\pi})^{3/2} \exp((\mu_A - m_A)/T) \,,$$

where $\mu_A$ is the chemical potential of the species. If the nuclear reactions that produce nucleus $A$ out of $Z$ protons and $A-Z$ neutrons occur rapidly compared to the expansion rate, chemical equilibrium also obtains.

The binding energy of the nuclear species $A(Z)$ is

$$B_A = Zm_p + (A-Z)m_n - m_A \,,$$

and the abundance of species $A(Z)$ is

$$n_A = g_A A^{3/2} 2^{-A} (\frac{2\pi}{m_p T})^{\frac{3(A-1)}{2}} {n_p}^Z {n_n}^{A-Z} \exp(\frac{B_A}{T}) \,.$$

At the onset of nucleosynthesis ($T >> 1$ MeV, $t << 1$ sec) the balance between neutrons and protons is maintained by the week interactions (here $\nu = \nu_e$):

$$n \leftrightarrow p + e^- + \bar{\nu}\,,$$

$$\nu + n \leftrightarrow p + e^-\,,$$

$$e^+ + n \leftrightarrow p + \bar{\nu}\,.$$

When the rates for these reaction are rapid compared to the expansion rate $H$, chemical equilibrium obtains,

$$\mu_n + \mu_\nu = \mu_p + \mu_e\,,$$

what implies that in chemical equilibrium

$$\frac{n_n}{n_p} = \exp(-Q/T + (\mu_e - \mu_\nu)/T)\,,$$

where $Q = m_n - m_p = 1.293$Mev. Neglecting the chemical potential, the equilibrium value of the neutron-to-proton ratio is

$$\frac{n_n}{n_p} = \exp(-Q/T)\,,$$