Introduction to Cosmology

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Relativistics cosmological models

Robertson and Walker showed that the line element of a 3 dimensional homogeneous and isotropic space can be always reduced to the following form:

$$dl^2 = \frac{dr^2}{1 - k\,r^2} + r^2\,(\,d\Theta^2 + \sin^2\Theta\;d\phi^2\,)$$

where

$$k = \begin{cases} +1 & \text{spherical space} \\ 0 & \text{flat space} \\ -1 & \text{hyperbolic space} \end{cases}$$

The Friedman model also known as Friedman-Lemaitre-Robertson-Walker model:

$$ds^2 = c^2 dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right)$$

R(t) – the scale factor
k – curvature parameter

The Einstein field equations

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{8\pi G}{c^2}p = -\frac{kc^2}{R^2} + \Lambda c^2 \,, \quad (1)$$

$$\frac{\dot{R}^2}{R^2} - \frac{8\pi G}{3}\varrho = -\frac{kc^2}{R^2} + \frac{1}{3}\Lambda c^2 \,, \quad (2)$$

subtracting the second equation from the first, we get

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\varrho + \frac{3p}{c^2}) + \frac{\Lambda c^2}{3} \,, \quad (3)$$

ρ – the density of matter (energy)
p – the pressure
Λ – the cosmological constant

some time ago we introduced the Hubble constant via $v = H \cdot d$,

it turns out that it is connected with the rate of change of the scale factor R(t)

$$H(t) = \frac{\dot{R}(t)}{R(t)} . \tag{4}$$

Let us introduce a so called deceleration parameter q(t), defined as

$$q(t) = -\frac{\ddot{R}R}{\dot{R}^2} , \tag{5}$$

equation (3) can be rewritten as

$$(\varrho + \frac{3p}{c^2}) - \frac{\Lambda c^2}{4\pi G} = \frac{3H^2 q}{4\pi G} , \tag{6}$$

equation (2) can be rewritten as

$$\frac{kc^2}{R^2} = \frac{1}{3}(8\pi G\varrho + \Lambda c^2) - H^2 , \tag{7}$$

Using equation (6) can be transformed into:

$$\frac{kc^2}{R^2} = \frac{4\pi G}{3q}(\varrho(2q-1) - \frac{3p}{c^2}) + \frac{\Lambda c^2}{3}(1 + \frac{1}{q}) . \tag{8}$$

How to determine H and q ?

Let us recall the relation $1+z=\frac{R(t_0)}{R(t_e)}$

$$1+z=\frac{R(t_0)}{R(t_0-\Delta t)}=1+\Delta\frac{\dot{R}(t_0)}{R(t_0)}+\Delta^2(\frac{\dot{R}_0^2}{{R_0}^2}-\frac{\ddot{R}_0}{2R_0}),$$

flux $l=\frac{L}{4\pi r^2 R^2{}_0(1+z)^2}$ so the luminosity distance $d_L=(\frac{L}{4\pi l})^{1/2}=rR_0(1+z)$,

for small z, $c\cdot z=H\cdot d_L$

In astronomy the distance-magnitude relation is usually used $m=5\log D+M-5$

Using the luminosity distance this relation can be transformed into (not easy!)

$$m-M=5\log\frac{cz}{H_0}+1.086(1-q_0)z-0.27(1-q_0)(1+3q_0)z^2+25$$

H_o $in\,\frac{km}{sMpc}$, c in km/s

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In[4]:= Plot[5 Log[3 10^8 x / 73] + 1.086 / 2 x - 0.27 5 / 4 x^2 + 25, {x, 0, 0.1}, AxesLabel → {z, μ}]
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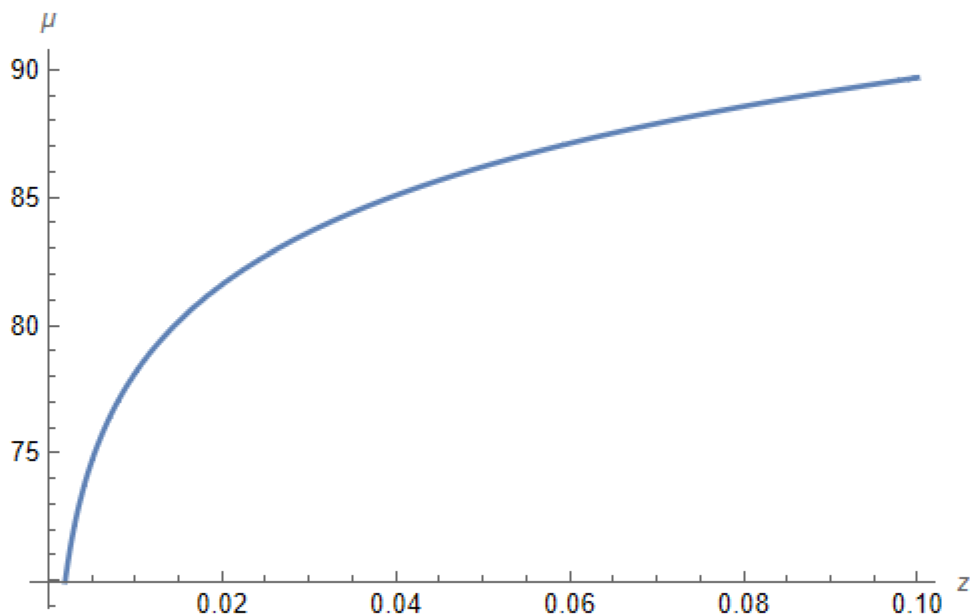
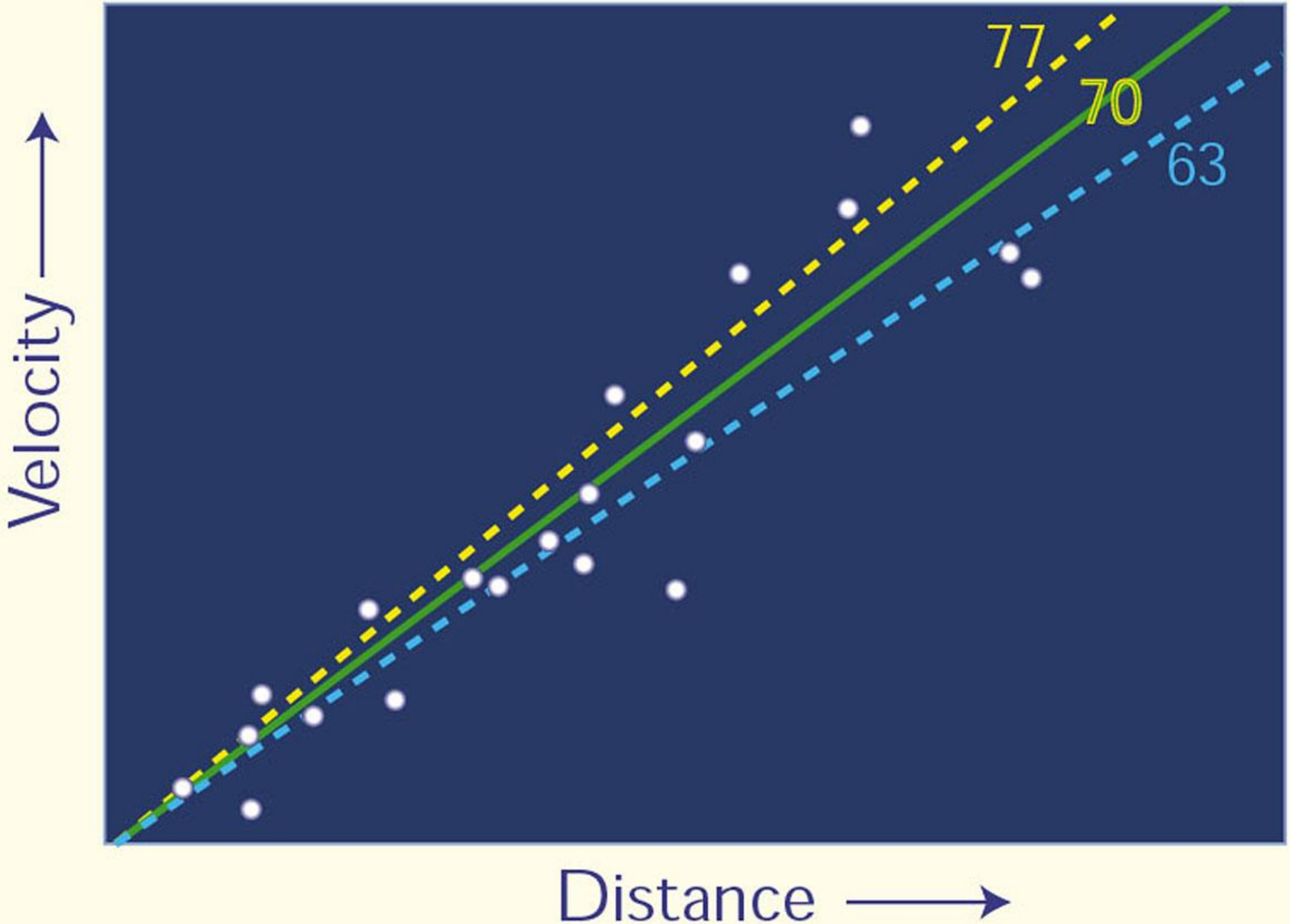
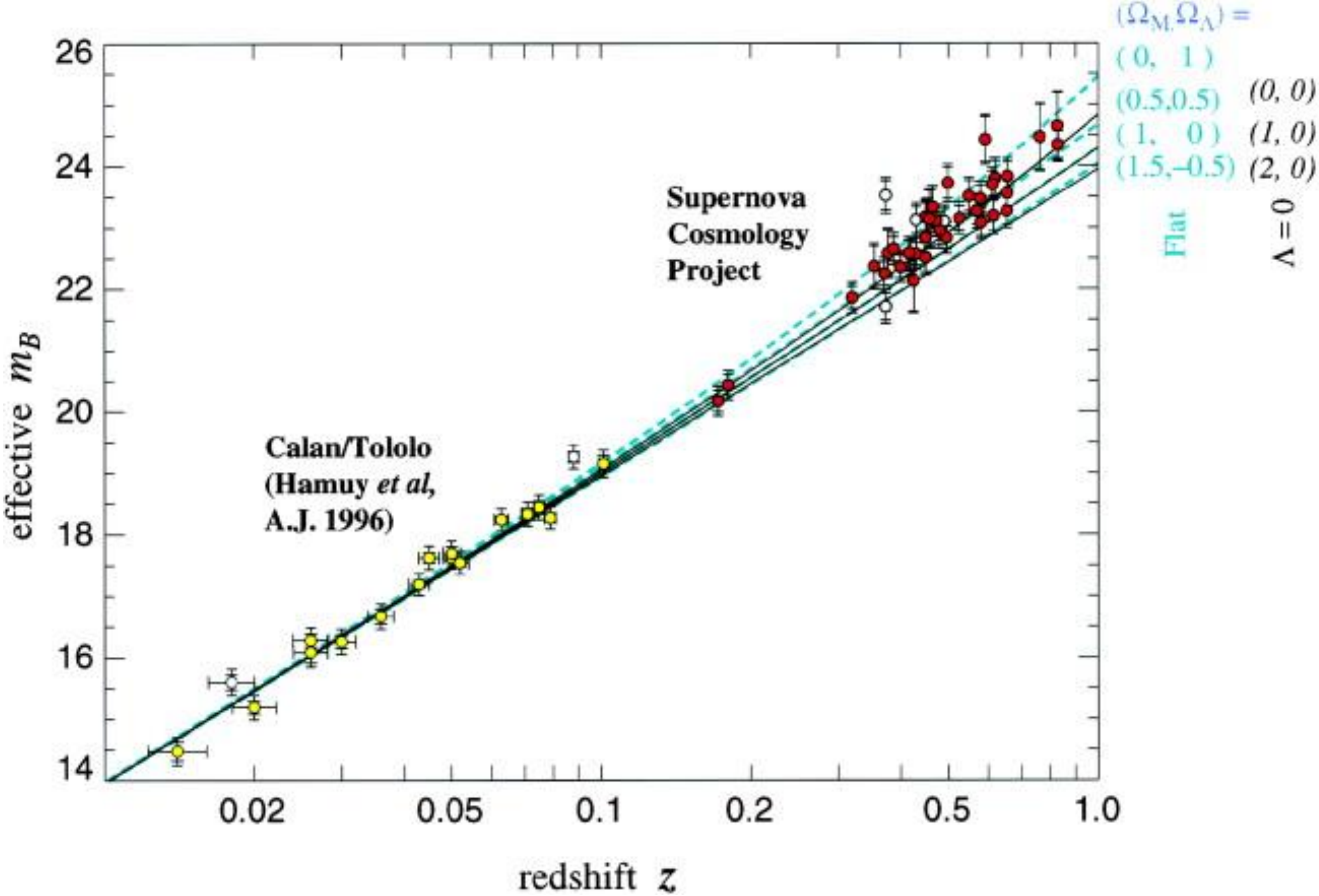
Out[4]=



Hubble Diagram for Cepheids





(ΩM,ΩΛ) =
(0, 1)
(0.5,0.5)
(1, 0)
(1.5,–0.5)
Flat
(0, 0)
(1, 0)
(2, 0)
Λ = 0
Supernova
Cosmology
Project
Calan/Tololo
(Hamuy et al,
A.J. 1996)
effective m_B
redshift z
26
24
22
20
18
16
14
0.02
0.05
0.1
0.2
0.5
1.0

Some exact solutions of the Friedman equations

Let us consider pressureless gas in a flat universe with $\Lambda = 0$

In this case the equation (2) reduces to

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\varrho \,. \tag{9}$$

The energy-momentum conservation law ${T^{ab}}_{;b} = 0$, reduces to:

$$\varrho \cdot R^3 = \mathrm{const} \,. \tag{10}$$

From equations (9) and (10), it follows that

$$R(t) \propto t^{2/3} \,.$$

In such matter dominated universe $H(t) = \dfrac{2}{3t}$ and $q = \frac{1}{2}$.

Let us discuss the first obvious consequences:

$R(t) \sim t^{2/3}$ implies that when $t \to 0$, $R(t) \to 0$

so, the Universe had a beginning !

Since $\varrho \cdot R^3$=const, when $R \to 0$, $\varrho \to \infty$!!!

Early in 1940-ties George Gamow realized that if the early Universe was very dense it was also very hot. So let us consider radiation dominated Universe.

Basic thermodynamical properties of radiation:

$$\varepsilon_{rad} = a \cdot T^4,\ a - \text{Stefan} - \text{Boltzmann constant},\ p_{rad} = \frac{1}{3}\varepsilon_{rad}\,. \tag{11}$$

From the energy-momentum conservation law it follows that:

$$\frac{d}{dt}(\varepsilon_{rad} R^3) + p\frac{d}{dt}(R^3) = 0,\ \text{so}$$

$$\frac{d}{dt}(\varepsilon_{rad} R^3) + \frac{1}{3}\varepsilon_{rad}\frac{d}{dt}(R^3) = 0 \to$$

$$\frac{d}{dt}(\varepsilon_{rad} \cdot R^4) = 0,\ \to \varepsilon_{rad} \cdot R^4 = \text{const, or } T \cdot R = \text{const}\,.$$

Equation (2) assumes now the form:

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\varepsilon_{rad}\,, \text{or}\ \frac{\dot{R}^2}{R^2} =\sim \frac{1}{R^4}\,,$$

what leads to:

$$R(t) \sim t^{1/2}\,.$$

It means that when $R(t) \to 0, \quad T(t) \to \infty$!!!

The early Universe was very dense and very hot !!!

In such radiation dominated universe $H(t) = \dfrac{1}{2t}$ and $q = 1$.

Finally let us consider the flat, empty Universe with $\Lambda \neq 0$.

In this case equation (2) assumes a simple form:

$$\frac{\dot{R}^2}{R^2} = \frac{1}{3}\Lambda c^2 \,,$$

that leads to an exponential solution:

$$R(t) \sim exp(\sqrt{\frac{\Lambda c^2}{3}} \cdot t)\,.$$

In such dark energy dominated universe $H(t) = \sqrt{\frac{\Lambda c^2}{3}}$ and $q = 1$.

So summarizing we have:

$$R(t) \sim \begin{cases} t^{2/3}\,, & \text{matter dominated universe}\,, \\ t^{1/2}\,, & \text{radiation dominated universe}\,, \\ exp(\sqrt{\frac{\Lambda c^2}{3}} \cdot t)\,, & \text{dark energy dominated universe}\,. \end{cases}$$

Critical density

$$\rho_{crit} = \frac{3\,H^2}{8\pi\,G}$$

$$\Omega = \frac{\rho}{\rho_{crit}}$$

$$\Omega = \begin{cases} > 1\ spherical\ Universe\ (closed) \\ = 1\ flat\ Universe\ (open) \\ < 1\ hyperbolic\ Universe\ (open) \end{cases}$$

The notion of critical density allows convenient parametrization of the Hubble constant:

$$H(z) = H_0\sqrt{\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda}\,,\ \text{where}$$

Ω_r - represents contribution of radiation, Ω_m - matter,
Ω_k - curvature, and Ω_Λ - cosmological constant or Dark Energy.

This relation implies that if $\Omega_r \neq 0$ the early evolution
of the Universe was dominated by radiation.

Since $H(z=0) = H_0$ we also have a constrain:

$$\Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda = 1$$

Current values:

$$\Omega_r = 2.47 \cdot 10^{-5} h^{-2},\ \Omega_m = 0.315 \pm 0.007,\ \Omega_\Lambda = 0.685 \pm 0.007\,.$$

Using the Friedman equations and the Hubble law it is possible to calculate how much time a light signal emitted at z needed to reach us

$$1+z=\frac{R(t_0)}{R(t_e)}, \text{ let us use a common convention } R(t_0)=1,\ R(t_e)=R(t)\,,$$

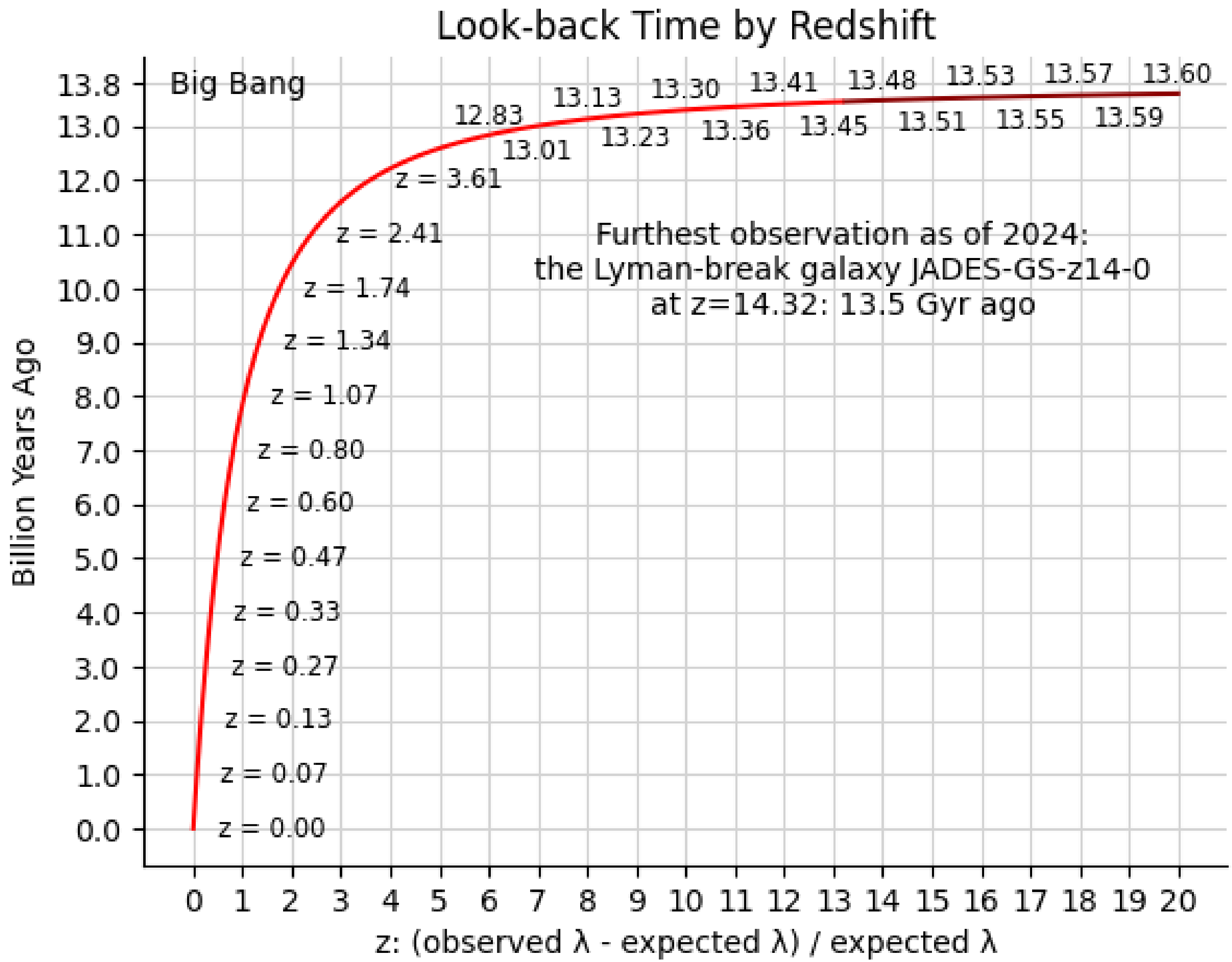
$$\frac{\dot{R}^2}{R^2}-\frac{8\pi G}{3}\varrho=-\frac{kc^2}{R^2}+\frac{1}{3}\Lambda c^2\,.$$

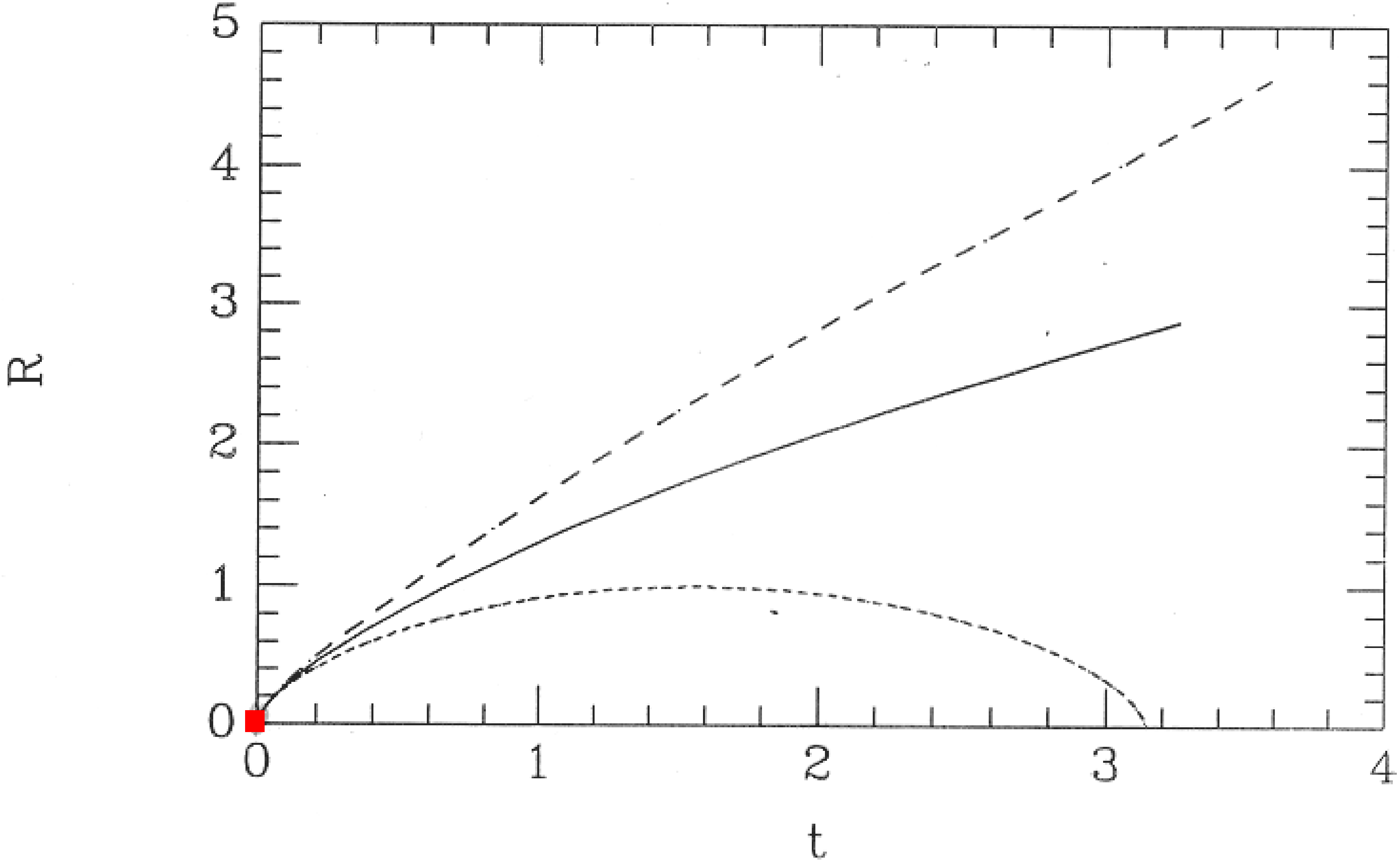
This equation can be rewritten as:

$$H(z)=H_0\sqrt{\Omega_r(1+z)^4+\Omega_m(1+z)^3+\Omega_k(1+z)^2+\Omega_\Lambda}\,.$$

Using $1+z=\dfrac{1}{R(t)}$, and $H=\dfrac{\dot{R}}{R}$, we find

$$t(z)=\frac{1}{H_0}\int_0^z\frac{dz}{(1+z)\sqrt{\Omega_r(1+z)^4+\Omega_m(1+z)^3+\Omega_k(1+z)^2+\Omega_\Lambda}}\,.$$


Look-back Time by Redshift
Big Bang
12.83
13.13
13.30
13.41
13.48
13.53
13.57
13.60
13.01
13.23
13.36
13.45
13.51
13.55
13.59
z = 3.61
z = 2.41
z = 1.74
z = 1.34
z = 1.07
z = 0.80
z = 0.60
z = 0.47
z = 0.33
z = 0.27
z = 0.13
z = 0.07
z = 0.00
Furthest observation as of 2024:
the Lyman-break galaxy JADES-GS-z14-0
at z=14.32: 13.5 Gyr ago
Billion Years Ago
13.8
13.0
12.0
11.0
10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.0
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
z: (observed λ - expected λ) / expected λ



R
t
0
1
2
3
4
5

The Universe started its evolution
from a state of infinite density
and infinite temperature
that we call the Big Bang

Penrose and Hawking singularity theorems

Under very general assumptions,
not requiring symmetries,
they have shown that the Universe
started its evolution from a state
of infinite density.

Big Bang is unavoidable !

singularity theorems://en.wikipedia.org/wiki/Penrose%E2%80%93Hawking_singularity_theorem

Energy scale:

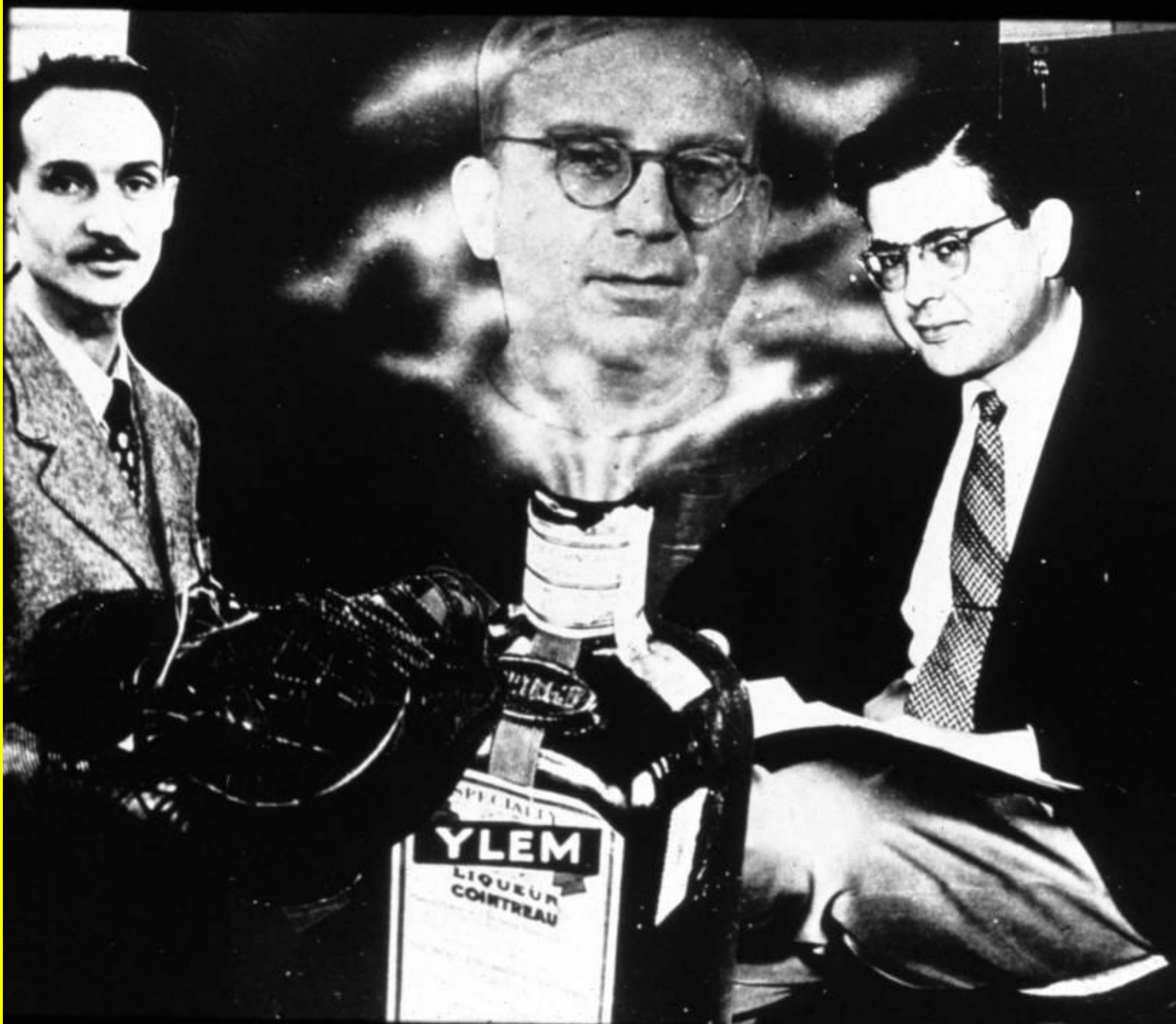
Energy: 1 Gev = $1.6022 \cdot 10^{-10}$ J

Temperature: 1 GeV = $1.605 \cdot 10^{13}$ K

Proton mass: 938.272 MeV

Neutron mass: 939.566 MeV

Electron mass: 0.5110 Mev


SPECIALTY
YLEM
LIQUEUR
COINTREAU

The state of thermal equilibrium

All particles, depending on their spins, are described by the Fermi-Dirac or Bose-Einstein phase space distribution:

$$f(\vec{p}) = [\exp((E-\mu)/T \pm 1]^{-1}\,,$$

where E - denotes energy, μ chemical potential, and the Boltzmann constant was set to be equal $k_B = 1$. The number density n, energy density ϱ and pressure p of a dilute, weakly interacting gas of particles with g internal degrees of freedom is given by;

$$n = \frac{g}{(2\pi)^3}\int f(\vec{p})d^3p\,,$$

$$\varrho = \frac{g}{(2\pi)^3}\int E(\vec{p})f(\vec{p})d^3p\,,$$

$$p = \frac{g}{(2\pi)^3}\int \frac{|\vec{p}|^2}{3E}f(\vec{p})d^3p\,,$$

where $E^2 = |\vec{p}|^2 + m^2$.

In kinetic equilibrium, the number density of a nonrelativistic nuclear species $A(Z)$ with mass number A and charge Z is given by

$$n_A = g_A(\frac{m_A T}{2\pi})^{3/2} \exp((\mu_A - m_A)/T)\,,$$

where μ_A is the chemical potential of the species. If the nuclear reactions that produce nucleus A out of Z protons and $A - Z$ neutrons occur rapidly compared to the expansion rate, chemical equilibrium also obtains.

The binding energy of the nuclear species $A(Z)$ is

$$B_A = Zm_p + (A - Z)m_n - m_A\,,$$

and the abundance of species $A(Z)$ is

$$n_A = g_A A^{3/2} 2^{-A}(\frac{2\pi}{m_p T})^{\frac{3(A-1)}{2}} {n_p}^Z {n_n}^{A-Z} \exp(\frac{B_A}{T})\,.$$

^{A}Z	B_A	g_A
^{2}H	2.22 MeV	3
^{3}H	6.92 MeV	2
^{3}He	7.72 MeV	2
^{4}He	28.3 MeV	1
^{12}C	92.2 MeV	1

At the onset of nucleosynthesis ($T >> 1$ MeV, $t << 1$ sec) the balance between neutrons and protons is maintained by the week interactions (here $\nu = \nu_e$):

$$n \leftrightarrow p + e^- + \bar{\nu} \,,$$

$$\nu + n \leftrightarrow p + e^- \,,$$

$$e^+ + n \leftrightarrow p + \bar{\nu} \,.$$

When the rates for these reaction are rapid compared to the expansion rate H, chemical equilibrium obtains,

$$\mu_n + \mu_\nu = \mu_p + \mu_e \,,$$

what implies that in chemical equilibrium

$$\frac{n_n}{n_p} = \exp(-Q/T + (\mu_e - \mu_\nu)/T) \,,$$

where $Q = m_n - m_p = 1.293$Mev. Neglecting the chemical potential, the equilibrium value of the neutron-to-proton ratio is

$$\frac{n_n}{n_p} = \exp(-Q/T) \,,$$

Composition of matter at $T \sim 0.2$ GeV

Photons, neutrinos, electrons and positrons, miuons, and tauons are in thermal equilibrium

$$\gamma + \gamma \leftrightarrow \mu^+ + \mu^- \leftrightarrow \nu_\mu + \tilde{\nu}_\mu \leftrightarrow \gamma + \gamma$$

$$\gamma + \gamma \leftrightarrow \tau^+ + \tau- \leftrightarrow \nu_\tau + \tilde{\nu}_\tau \leftrightarrow \gamma + \gamma$$

$$\gamma + \gamma \leftrightarrow e^+ + e^- \leftrightarrow \nu_e + \tilde{\nu}_e \leftrightarrow \gamma + \gamma$$

Reactions between leptons, like

$$e^+ + \mu^- \leftrightarrow \tilde{\nu}_e + \nu_\mu\,,\; e^- + \mu^+ \leftrightarrow \nu_e + \tilde{\nu}_\mu\,,$$

$$e^+ + \nu_e \leftrightarrow \mu^+ + \nu_\mu\,,\; e^- + \tilde{\nu}_e \leftrightarrow \mu_- + \tilde{\nu}_\mu\,.$$

preserve the state of thermal equilibrium.

Reaction rates between different leptons are determined by:

$$\Gamma_{e\nu} = n_e c \sigma_{e\nu} \quad \Gamma_{\mu\nu} = n_\mu c \sigma_{\mu\nu}\ \text{, where}$$

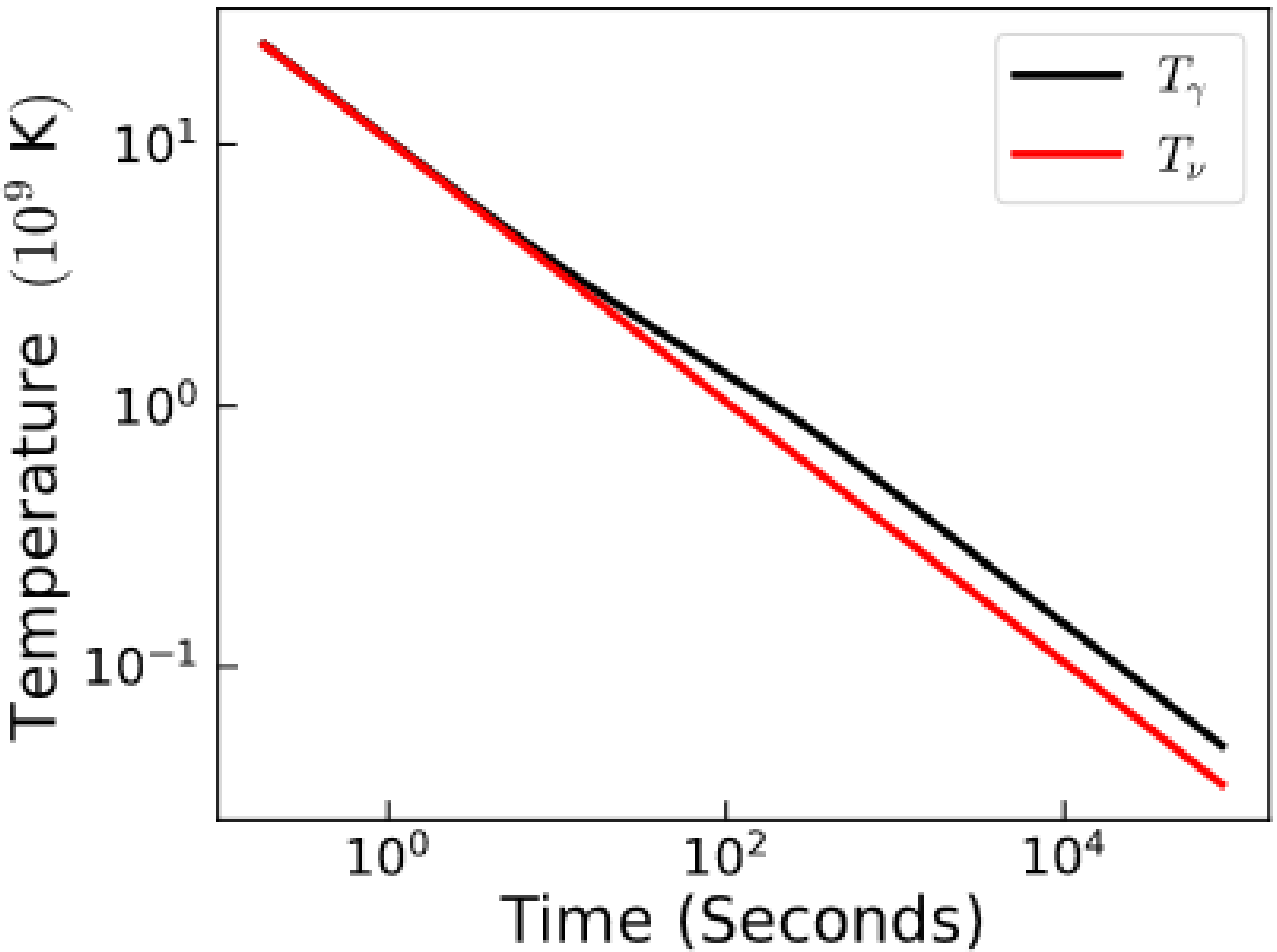
$\sigma_{e\nu}$ and $\sigma_{\mu\nu}$ denote the cross sections for the appropriate reactions. The state of thermal equilibrium is maintained when $\Gamma(T) \cdot t(T) \gg 1$. When this condition is violated, reactions with neutrinos are too slow to maintain thermal equilibrium.

At the moment of freeze out, the number density of neutrinos is $n_\nu = \frac{3}{8} g_\nu n_\gamma$, where $g_\nu = \Sigma g_{\nu i}$ denotes the total number of spin states.
At that epoch neutrons and protons are still in a state of thermal equilibrium, that is kept due to the reactions:

$$e_+ + n \leftrightarrow p + \tilde{\nu}_e \,,$$

$$\nu_e + n \leftrightarrow p + e^- \,,$$

$$n \leftrightarrow p + e^- + \tilde{\nu}_e \,.$$



T_γ
T_ν
Temperature $(10^9$ K)
10^1
10^0
10^{-1}
10^0
10^2
10^4
Time (Seconds)

However at certain temperature $T_* \approx 1 MeV$, $\Gamma(T_*) \cdot t(T_*) \approx 1$ and the number density of neutrons relative to protons becomes frozen at a level of:

$$\frac{n_n}{n_p} \approx (\frac{n_n}{n_p})_* = \exp(-\frac{\Delta mc^2}{k_B T_*}) \approx 0.27\,.$$

However free neutrons are unstable and they start to decay, so at onset of the primordial nucleosynthesis $\frac{n_n}{n_p} \approx 0.14$

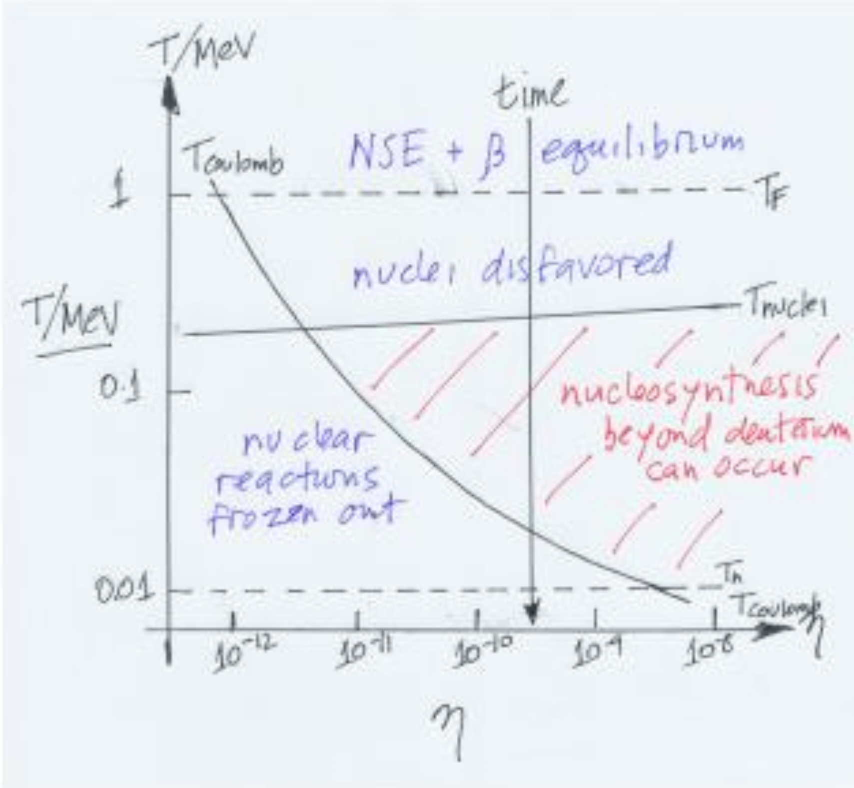
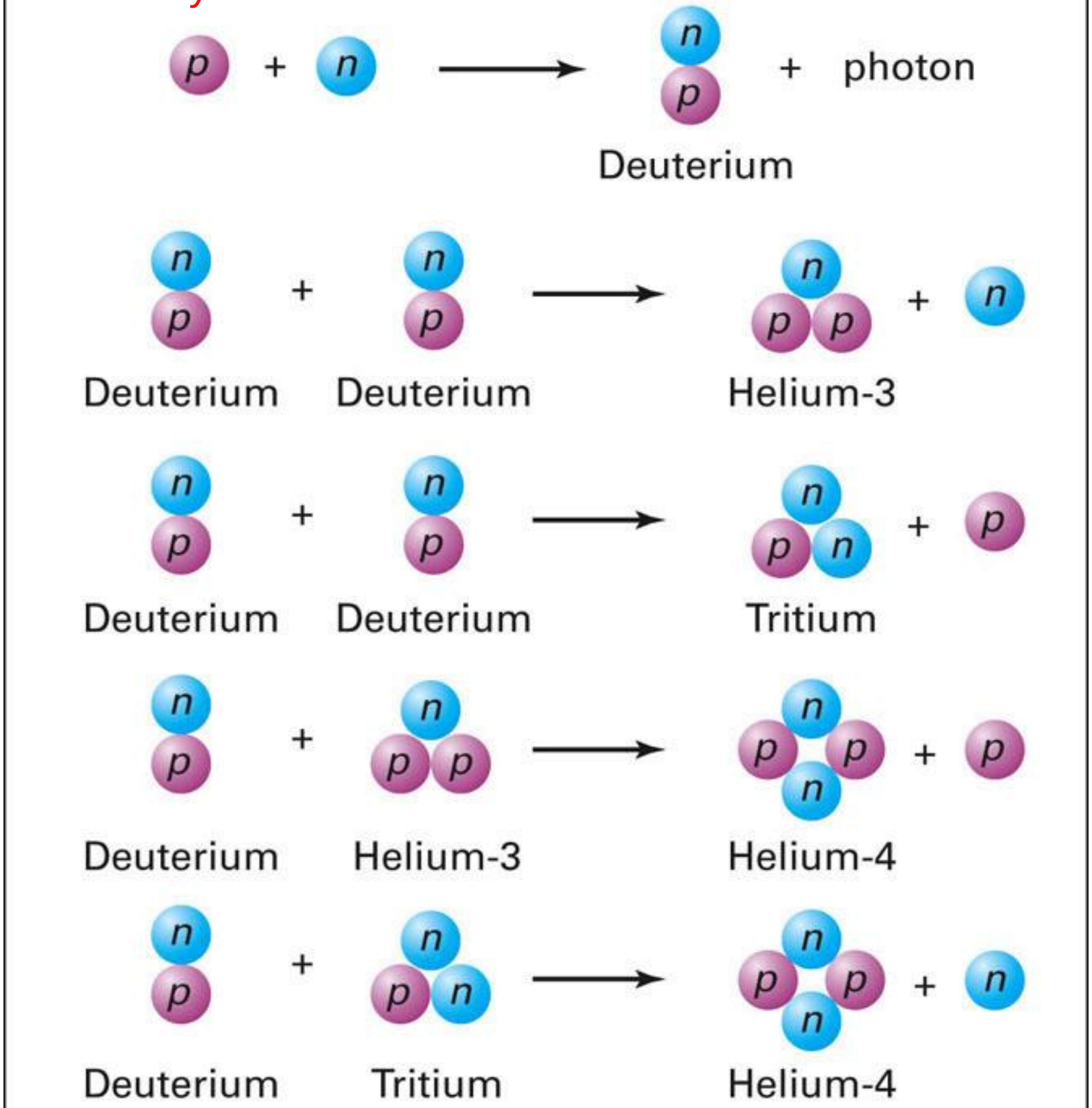


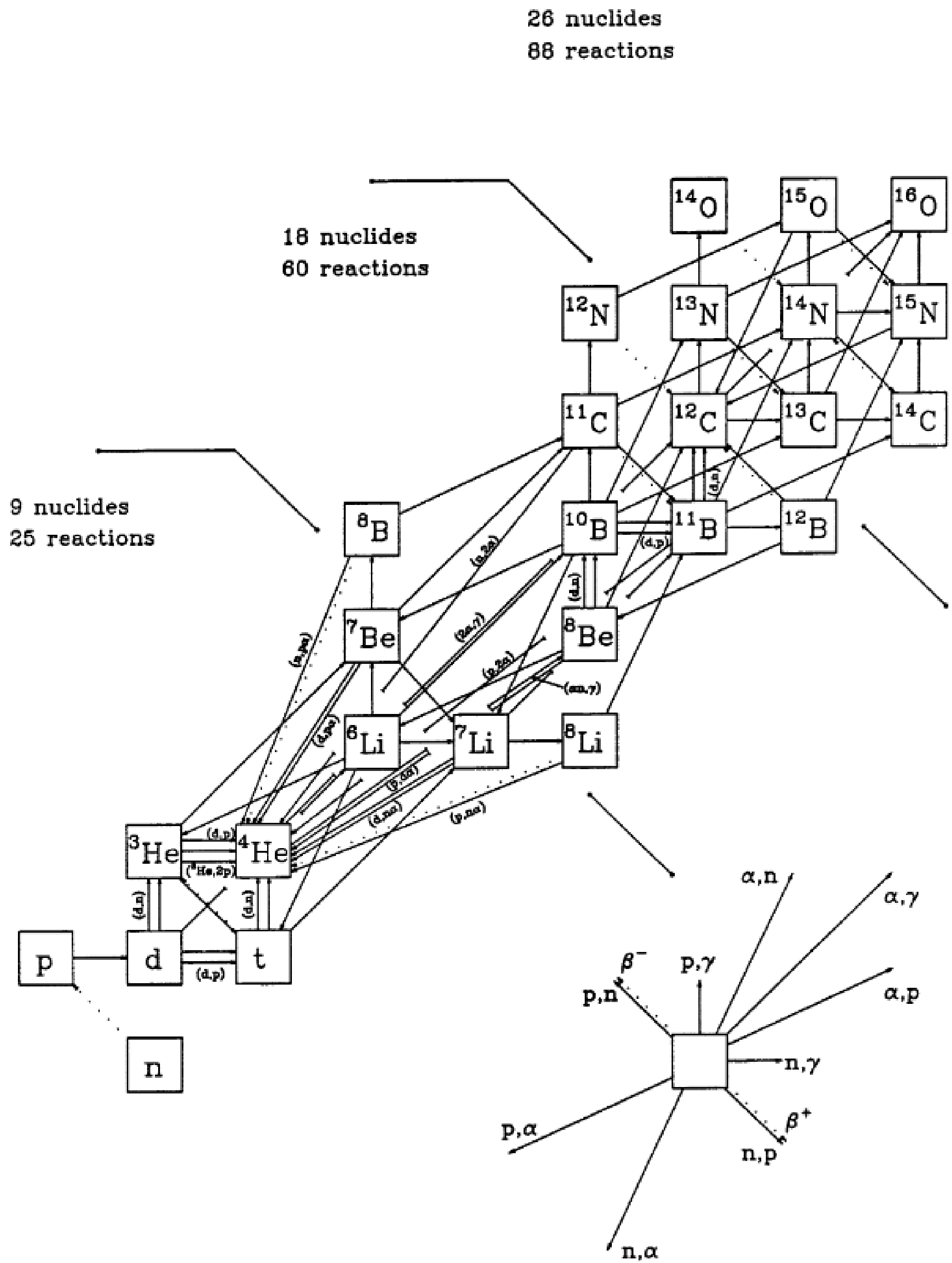
Figure 1: Key temperatures during BBN vs. the baryon-to-photon ratio η: T_F, the freeze in of the n/p ratio; $T_{\rm nuclei}$, the temperature below which nuclei are thermodynamically favored over free nucleons; $T_{\rm Coulomb}$, the temperature below which charged-particle nuclear reactions cease occurring; and T_n, the temperature at which the age of the Universe is the lifetime of a free neutron. Any significant nucleosynthesis beyond deuterium requires $T_{\rm nuclei} \geq T_{\rm Coulomb}$, or $\eta \geq 10^{-11}$. The vertical line marked "time" shows the timeline of successful BBN: freeze in of the n/p ratio at $T = T_F \longrightarrow$ period of waiting until nuclei are favored $T_F > T > T_{\rm nuclei} \longrightarrow$ nucleosynthesis $T_{\rm nuclei} > T > T_{\rm Coulomb} \longrightarrow$ frozen out nuclear reactions $T_{\rm Coulomb} > T \longrightarrow$ any free neutrons remaining decay $T_n > T$.

Nucleosynthesis in the first few minutes of the universe

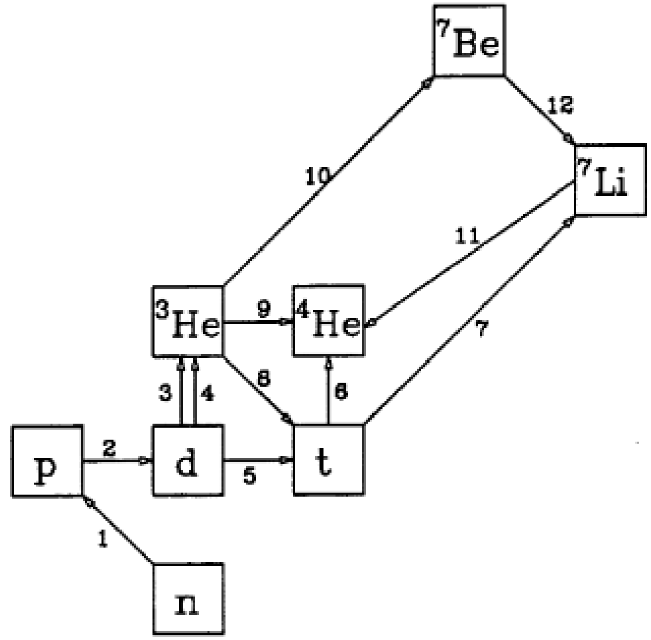


(After Alan Guth (MIT))

Temperature		Time
$10^{11} K$	—	0.01secs
$10^{10} K$	—	1.07secs
$10^{9} K$	—	3 mins
10^{8} K	—	5.3 hrs
$4 \times 10^{3} K$	—	10^{5} yrs



26 nuclides
88 reactions
18 nuclides
60 reactions
9 nuclides
25 reactions
14O
15O
16O
12N
13N
14N
15N
11C
12C
13C
14C
8B
10B
11B
12B
7Be
8Be
6Li
7Li
8Li
3He
4He
p
d
t
n
(d,n)
(d,p)
(p,nα)
(d,nα)
α,n
α,γ
p,γ
β⁻
p,n
α,p
n,γ
β⁺
n,p
p,α
n,α



1. n <–> p
2. p(n,γ)d
3. d(p,γ)^{3}He
4. d(d,n)^{3}He
5. d(d,p)t
6. t(d,n)^{4}He
7. t(α,γ)^{7}Li
8. ^{3}He(n,p)t
9. ^{3}He(d,p)^{4}He
10. ^{3}He(α,γ)^{7}Be
11. ^{7}Li(p,α)^{4}He
12. ^{7}Be(n,p)^{7}Li

Computational parameters:

cy = 0.300/ ct = 0.030/ initial temp = 1.00E+02/ final temp = 1.00E-02

smallest abundances allowed = 1.00E-25

Model parameters:

g = 1.00/ tau = 888.54/ # nu = 3.00/ lambda = 0.000E+00

xi-e = 0.000E+00/ xi-m = 0.000E+00/ xi-t = 0.000E+00

Temp	D/H	T/H	He3/H	He4	Li7/H
1.000E+02	3.724E-12	1.861E-25	1.861E-25	4.000E-25	1.861E-25
4.676E+01	1.447E-12	1.645E-24	1.878E-24	4.000E-25	1.724E-25
1.953E+01	6.514E-13	1.349E-24	1.771E-24	4.000E-25	1.483E-25
8.459E+00	6.931E-13	3.474E-23	4.290E-23	3.847E-23	1.284E-25
4.328E+00	3.435E-12	1.573E-19	9.482E-20	1.958E-17	1.216E-25
2.604E+00	6.453E-11	7.236E-14	1.227E-14	1.158E-12	1.197E-25
1.618E+00	9.186E-09	1.271E-09	2.918E-11	1.718E-08	7.021E-25
1.160E+00	2.277E-06	2.007E-07	5.884E-10	9.206E-07	3.093E-18
9.468E-01	2.099E-04	5.488E-06	7.423E-09	1.886E-04	2.221E-13
8.224E-01	4.693E-03	9.150E-05	2.299E-06	7.544E-02	1.505E-09
6.152E-01	3.355E-04	3.918E-06	1.305E-05	2.414E-01	3.726E-10
2.704E-01	8.010E-05	3.642E-07	1.401E-05	2.419E-01	6.245E-11
1.232E-01	7.319E-05	2.464E-07	1.507E-05	2.419E-01	6.051E-11
7.135E-02	7.268E-05	2.340E-07	1.521E-05	2.419E-01	6.066E-11
2.987E-02	7.261E-05	2.345E-07	1.523E-05	2.419E-01	6.067E-11
1.247E-02	7.260E-05	2.342E-07	1.523E-05	2.419E-01	6.067E-11
9.882E-03	7.260E-05	2.339E-07	1.547E-05	2.394E-01	1.271E-10

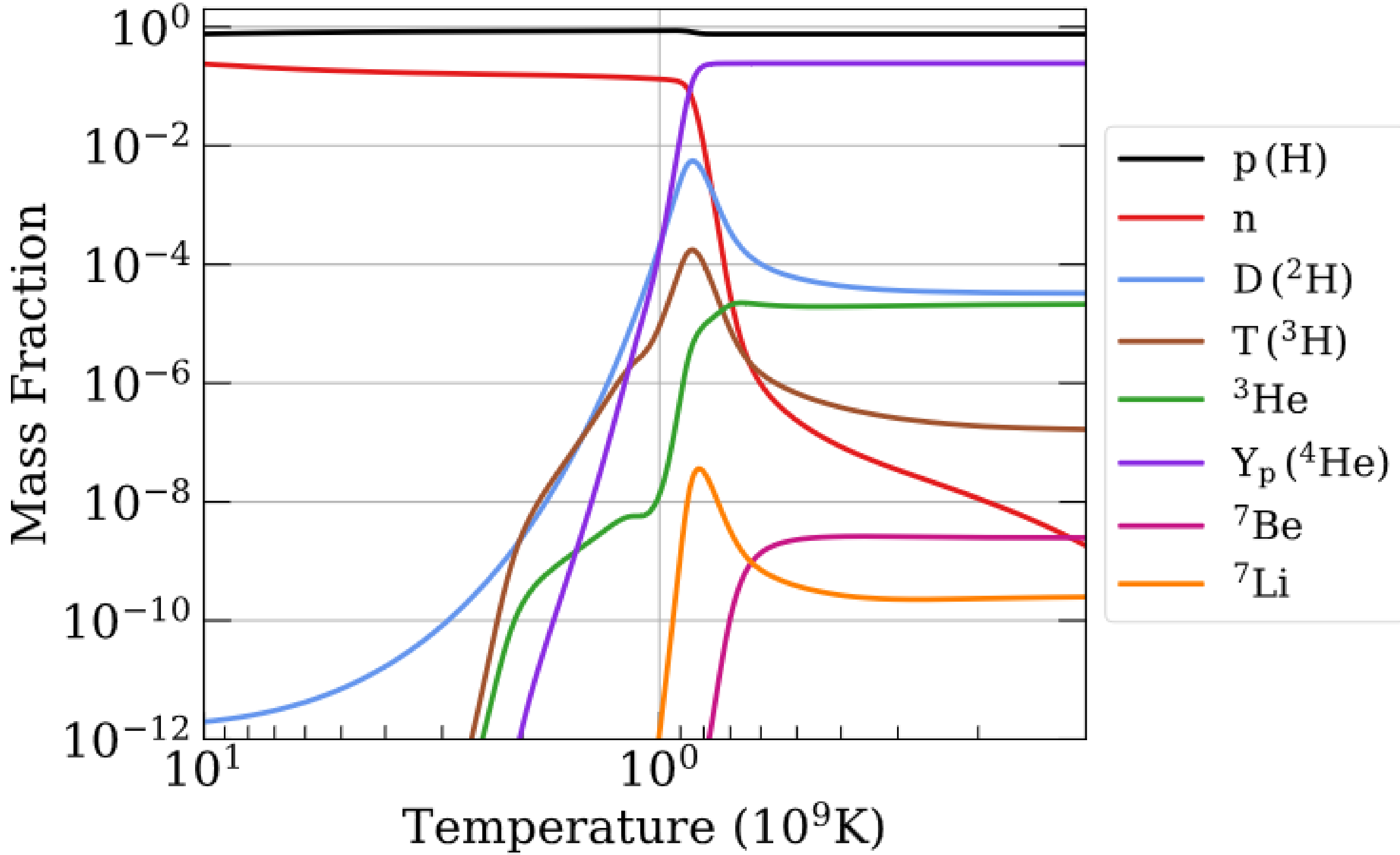
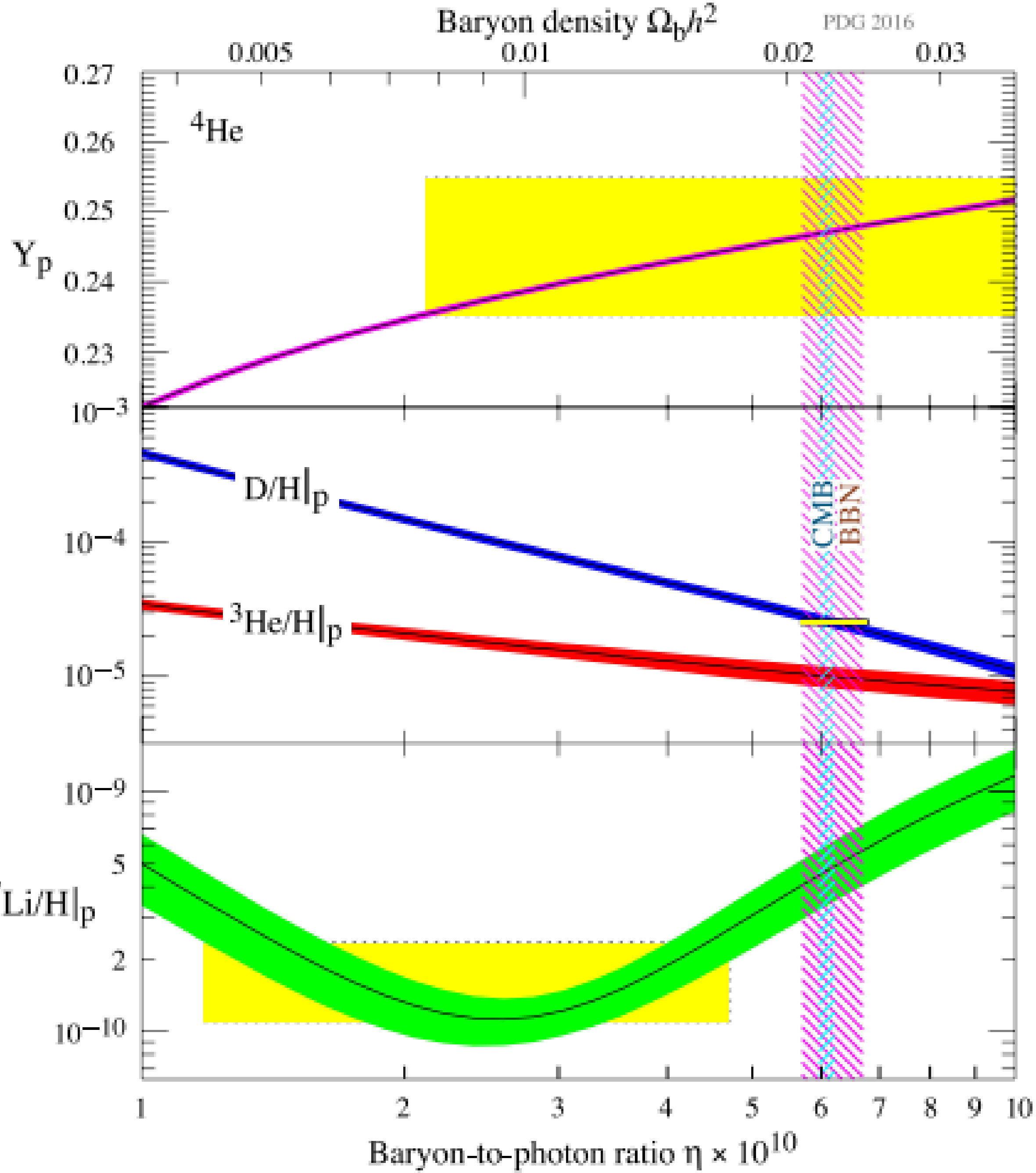


FIG. 3. Abundances of the elements created during BBN, produced using `BBN-simple`. For this calculation we adopt the baryon-to-photon ratio $\eta = 6.12 \times 10^{-10}$, the effective number of relativistic species $N_{\rm eff} = 3.046$, and the mean lifetime of the neutron, $\tau_n \approx 880.2$ s. See text for other details.



Baryon density $\Omega_b h^2$
PDG 2016
0.005
0.01
0.02
0.03
0.27
0.26
0.25
0.24
0.23
Y_p
^{4}He
10^{-3}
10^{-4}
10^{-5}
D/H|p
^{3}He/H|p
CMB
BBN
10^{-9}
5
2
10^{-10}
^{7}Li/H|p
1
2
3
4
5
6
7
8
9
10
Baryon-to-photon ratio $\eta \times 10^{10}$

Primordial nucleosythesis - observational data

$$D/H|_p = (2.53 \pm 0.04) \times 10^{-5}.$$

$$Y_p = 0.245 \pm 0.004,$$

$$Li/H|_p = (1.6 \pm 0.3) \times 10^{-10}.$$

$$5.8 \leq \eta_{10} \leq 6.6 \text{ (95\% CL)}.$$

$$0.021 \leq \Omega_b h^2 \leq 0.024 \text{ (95\% CL)},$$