Introduction to Cosmology

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Geometry

Axiomatic approach - Euclid

Analytic approach - Descartes

3 dimensional flat geometry, Cartesian coordinates

line element (metric) $dl^2 = dx^2 + dy^2 + dz^2$

spherical coordinates

$$x = r \sin\theta \cos\varphi \,,$$

$$y = r \sin\theta \sin\varphi \,,$$

$$z = r \cos\theta \,,$$

line element

$$dl^2 = dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \,,$$

Curved spaces

line element $dl^2 = g_{ab}(x^c)dx^a dx^b$, $g_{ab} = g_{ba}$

It is not possible to reduce the metric tensor g_{ab} to the identity matrix by a coordinate transformation

Geodesics:

how to find a line of shortest distance joining P and Q

variational principle:

$$\int_P^Q \sqrt{g_{ab} \frac{dx^a}{dl} \frac{dx^b}{dl}}\, dl$$

$$L = g_{ab} \frac{dx^a}{dl} \frac{dx^b}{dl}$$

Lagrange equations:

$$\frac{d}{dl}\frac{\partial L}{\partial \frac{dx^c}{dl}} - \frac{\partial L}{\partial x^c} = 0$$

$$2\frac{d}{dl}\left(g_{ac}\frac{dx^a}{dl}\right) - g_{ab,c}\frac{dx^a}{dl}\cdot\frac{dx^b}{dl} = 0$$

$$2\left(g_{ac,b}\frac{dx^b}{dl}\cdot\frac{dx^a}{dl} + g_{ac}\frac{d^2x^a}{dl^2}\right) - g_{ab,c}\frac{dx^a}{dl}\frac{dx^b}{dl} = 0$$

$$2g_{ac}\frac{d^2x^a}{dl^2} + \left(g_{ac,b} + g_{bc,a} - g_{ab,c}\right)\frac{dx^a}{dl}\cdot\frac{dx^b}{dl} = 0 \quad \Big| \frac{g^{cd}}{2}$$

$$g^{cd} g_{ac} \frac{d^2x^a}{dl^2} + \frac{1}{2} g^{cd} (g_{ac,b} + g_{bc,a} - g_{ab,c}) \frac{dx^a}{dl} \cdot \frac{dx^b}{dl} = 0$$

$$g^{cd} g_{ac} = \delta^d_c = I \qquad \frac{d^2x^d}{dl^2} + \frac{1}{2} g^{cd} (g_{ac,b} + g_{bc,a} - g_{ab,c}) \frac{dx^a}{dl} \frac{dx^b}{dl} = 0$$

$$\frac{d^2x^d}{dl^2} + \Gamma^d_{ab} \frac{dx^a}{dl} \frac{dx^b}{dl} = 0 \qquad \Gamma^d_{ab} = \Gamma^d_{ba}$$ – christoffel symbol

$$\Gamma^d_{ab} = \frac{1}{2} g^{dc} (g_{ac,b} + g_{bc,a} - g_{ab,c})$$

$\Gamma^d{}_{ab} = \Gamma^d{}_{ba}$

Covariant derivative

$$\frac{d^2x^d}{dl^2} + \Gamma^d_{ab}\frac{dx^a}{dl}\cdot\frac{dx^b}{dl} = 0$$

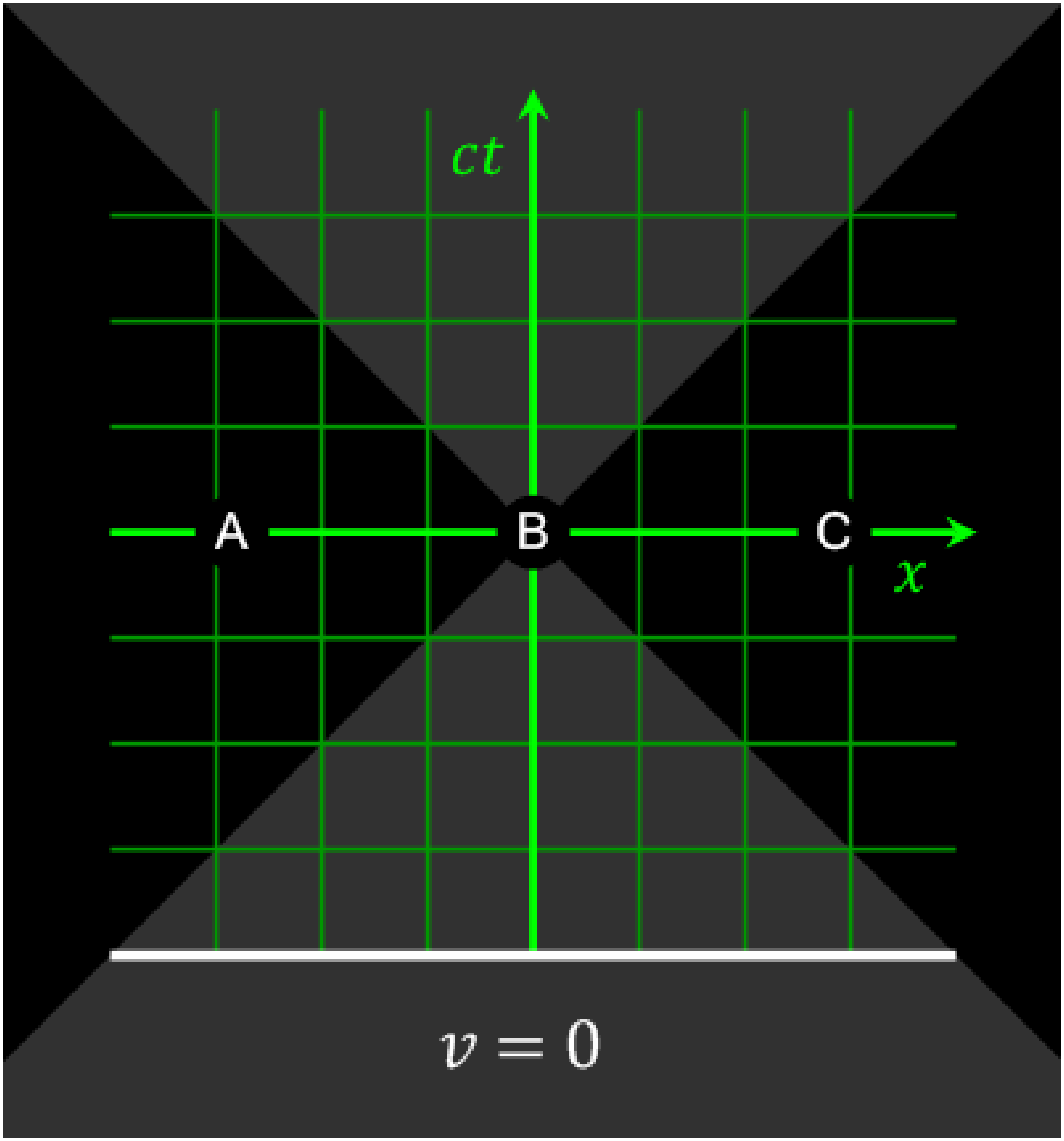
$$\frac{dx^a}{dl} = t^a \qquad t^a{}_{;b}\, t^b = 0$$

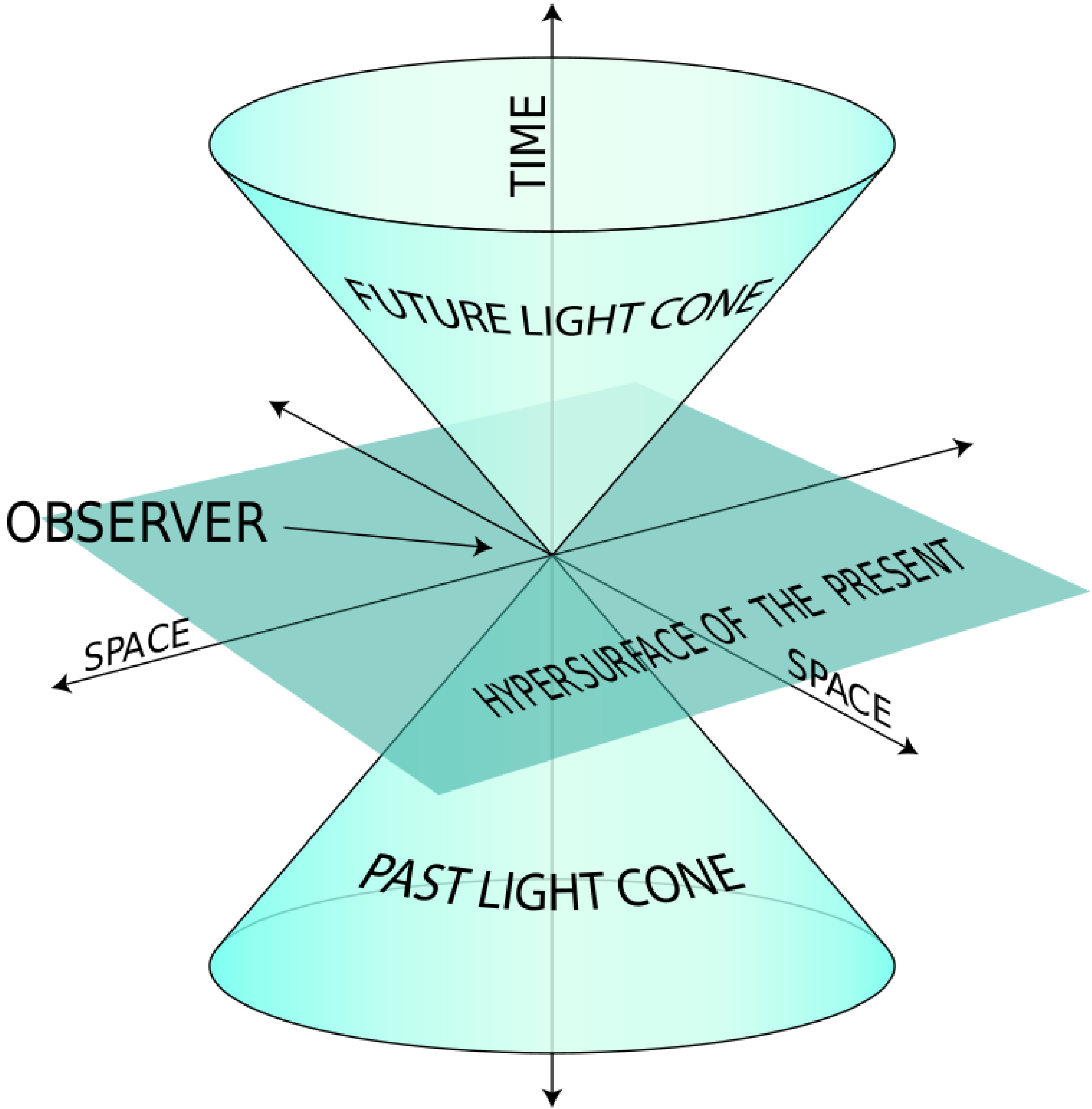
$$g^{ab}{}_{;c} = 0$$

The metric tensor is covariantly constant !!!

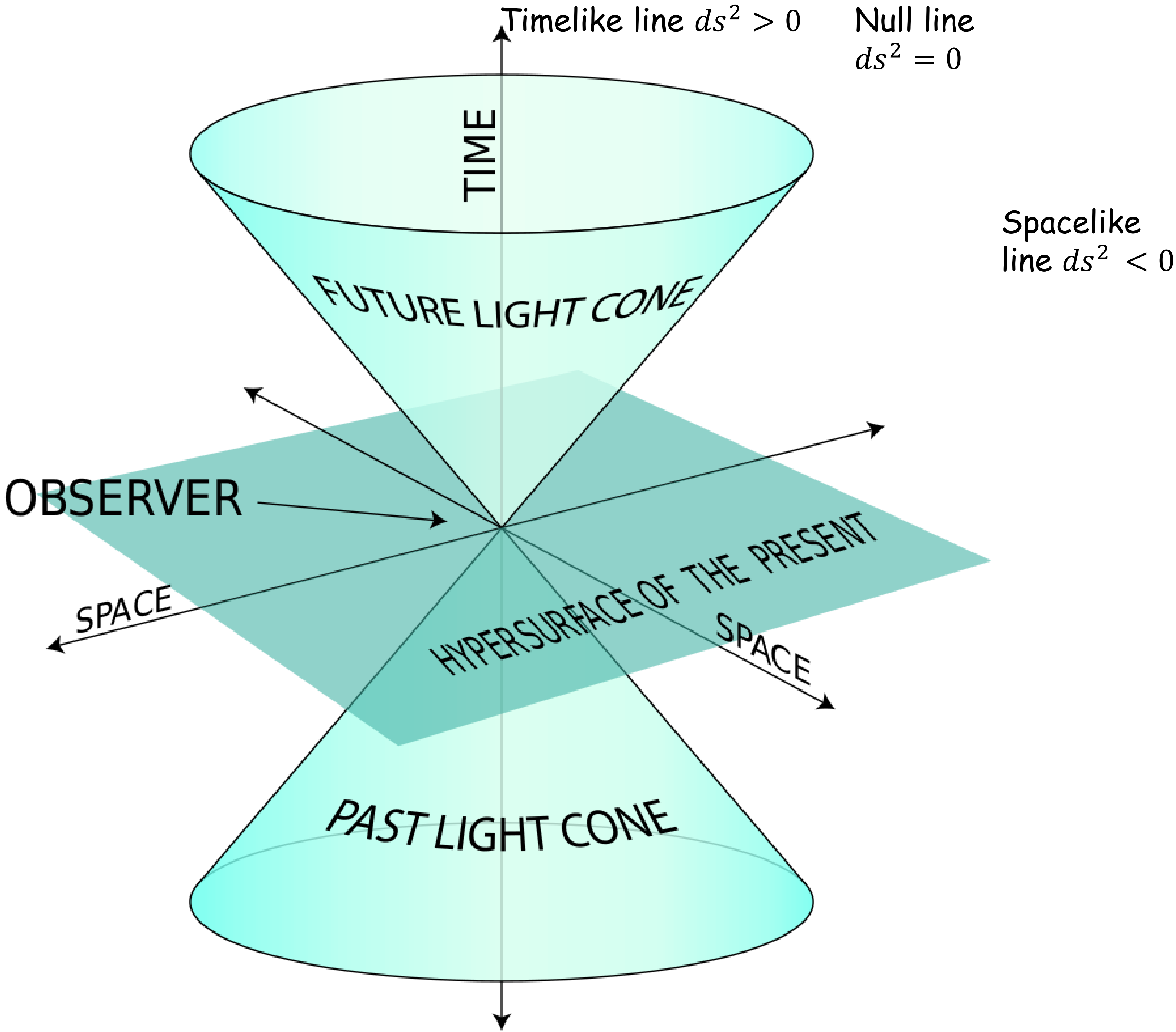
The Minkowski space-time

$$ds^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2)$$


ct
A
B
C
x
v = 0



TIME
FUTURE LIGHT CONE
OBSERVER
HYPERSURFACE OF THE PRESENT
SPACE
SPACE
PAST LIGHT CONE


Timelike line $ds^2 > 0$
Null line
$ds^2 = 0$
Spacelike
line $ds^2 < 0$
TIME
FUTURE LIGHT CONE
OBSERVER
SPACE
HYPERSURFACE OF THE PRESENT
SPACE
PAST LIGHT CONE

$$E = m \cdot c^2$$

Die Grundlage der allgemeinen Relativitätstheorie

von

A. Einstein

Leipzig :: Verlag von Johann Ambrosius Barth :: 1916

NEWTON
Absolute space
Absolute time

Einstein
Special theory of relativity
Spacetime is absolute

Einstein
General theory of relativity
Properties of spacetime depend on the distribution of matter

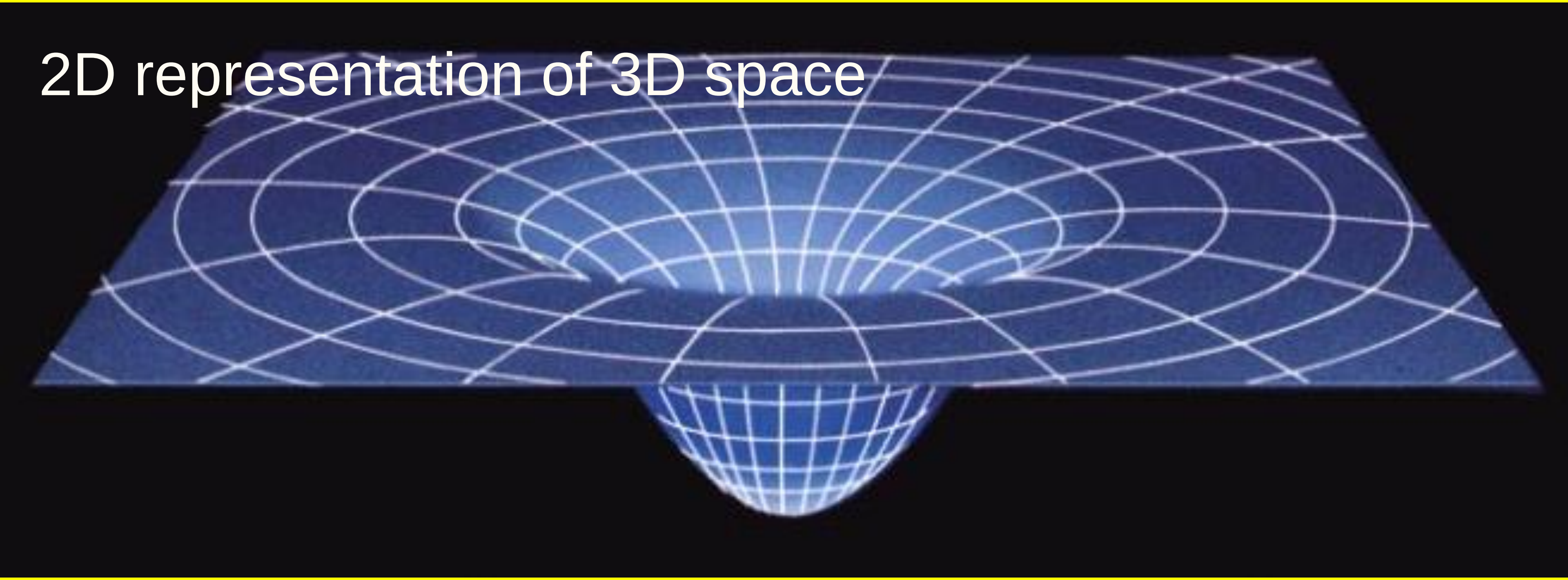
Distribution of matter <=> geometry of spacetime

Einstein's equations

Curvature of spacetime = κ density of matter

$\kappa \simeq 2 \cdot 10^{-43}$ in SI units

Warping of Space



gravity deforms space-time

Light travels along “straight” lines in a curved space-time

Geometry of spacetime is determined by
the distribution of matter

$$R^{i}{}_{klm} = \frac{\partial \Gamma^{i}_{km}}{\partial x^{l}} - \frac{\partial \Gamma^{i}_{kl}}{\partial x^{m}} + \Gamma^{i}_{nl}\Gamma^{n}_{km} - \Gamma^{i}_{nm}\Gamma^{n}_{kl}.$$

The Riemann curvature tensor

$$R_{ik} = g^{lm} R_{limk} = R^{l}_{ilk}.$$

The Ricci tensor

$$R_{ik} = \frac{\partial \Gamma^{l}_{ik}}{\partial x^{l}} - \frac{\partial \Gamma^{l}_{il}}{\partial x^{k}} + \Gamma^{l}_{ik}\Gamma^{m}_{lm} - \Gamma^{m}_{il}\Gamma^{l}_{km}.$$

$$R = g^{ik} R_{ik} = g^{il} g^{km} R_{iklm},$$

The Ricci scalar

The Einstein equations

$$R_{ij} - \frac{1}{2} g_{ij} R = \frac{8\pi G}{c^4} T_{ij} + \Lambda g_{ij}$$

G - the gravitational constant
c - velocity of light
Λ - the cosmological constant

$$T^{ij}{}_{;j} = 0$$

$$T^{ij} = (\varepsilon + p) u^i u^j - p g^{ij}$$

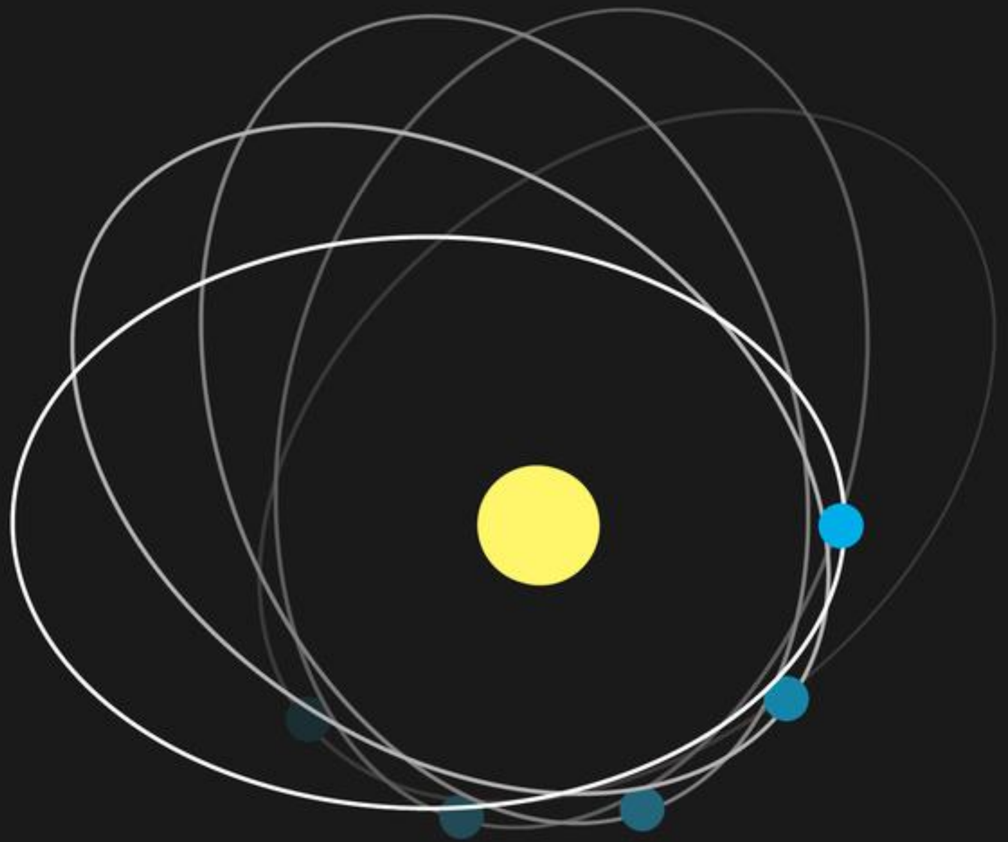
ε - the energy density
p - the pressure

Basic predictions of Einstein's general theory of relativity:

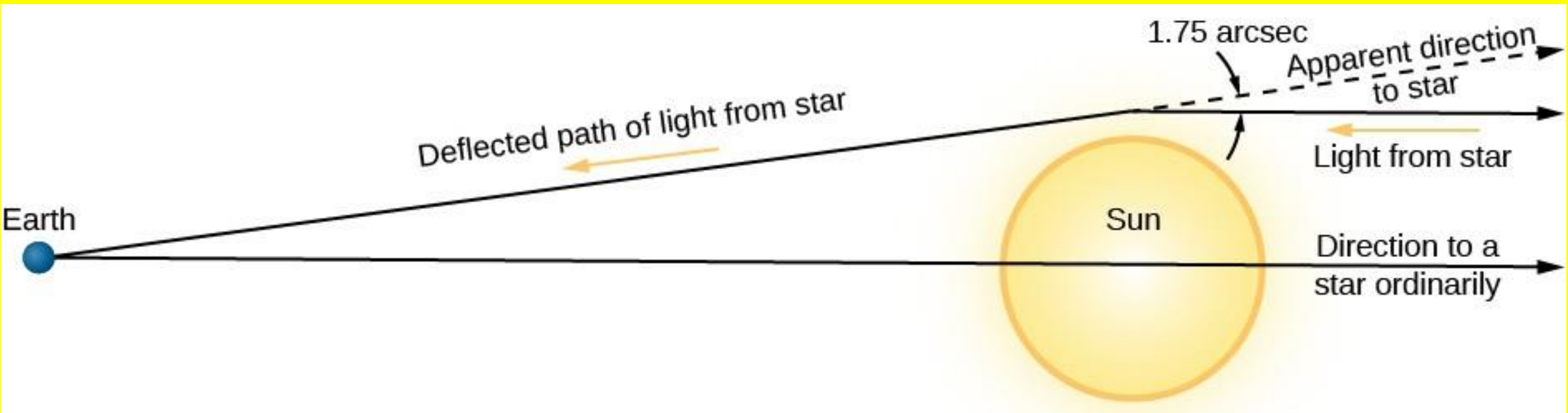
Perihelion motion

Bending of light

Gravitational redshift



43” per century

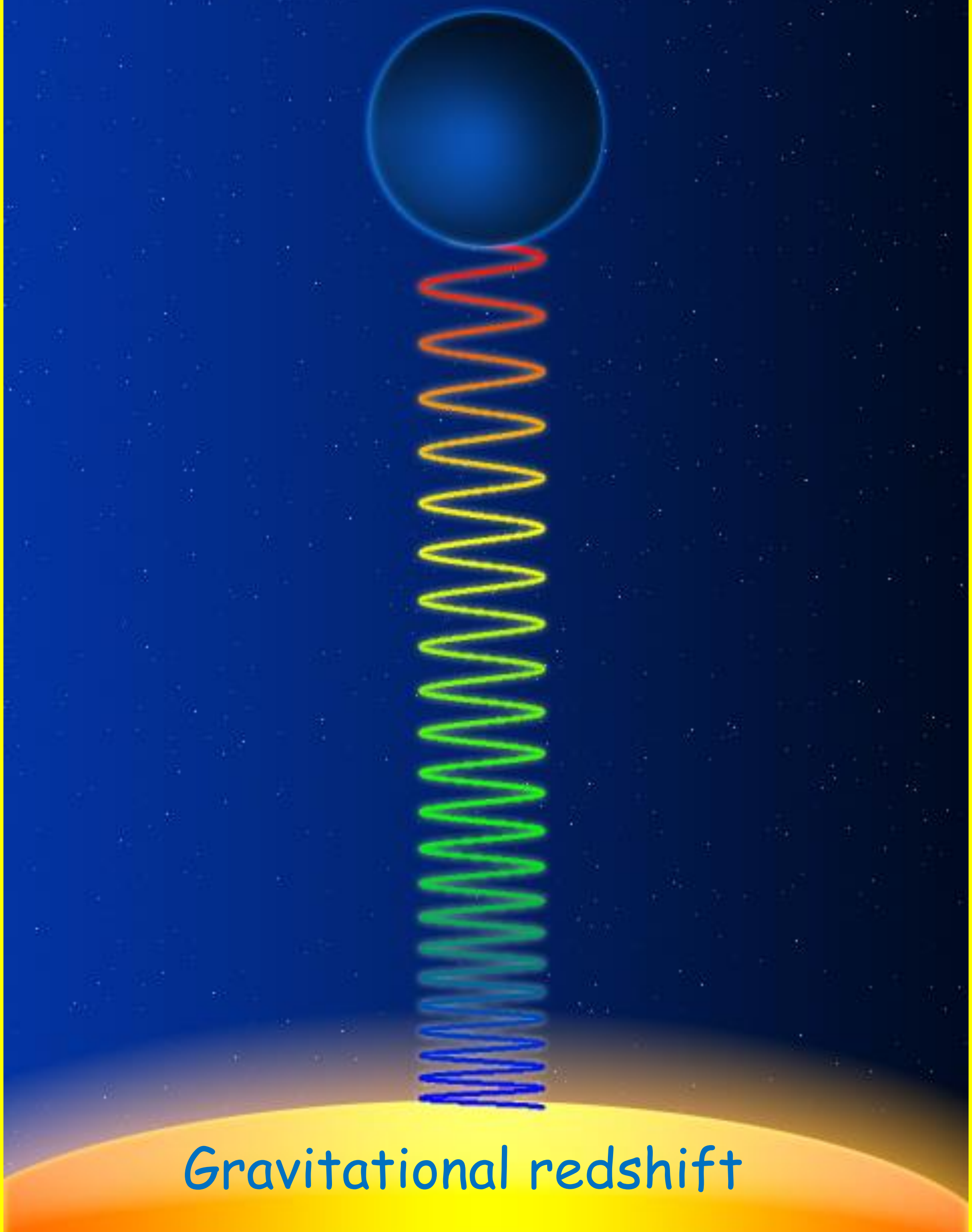


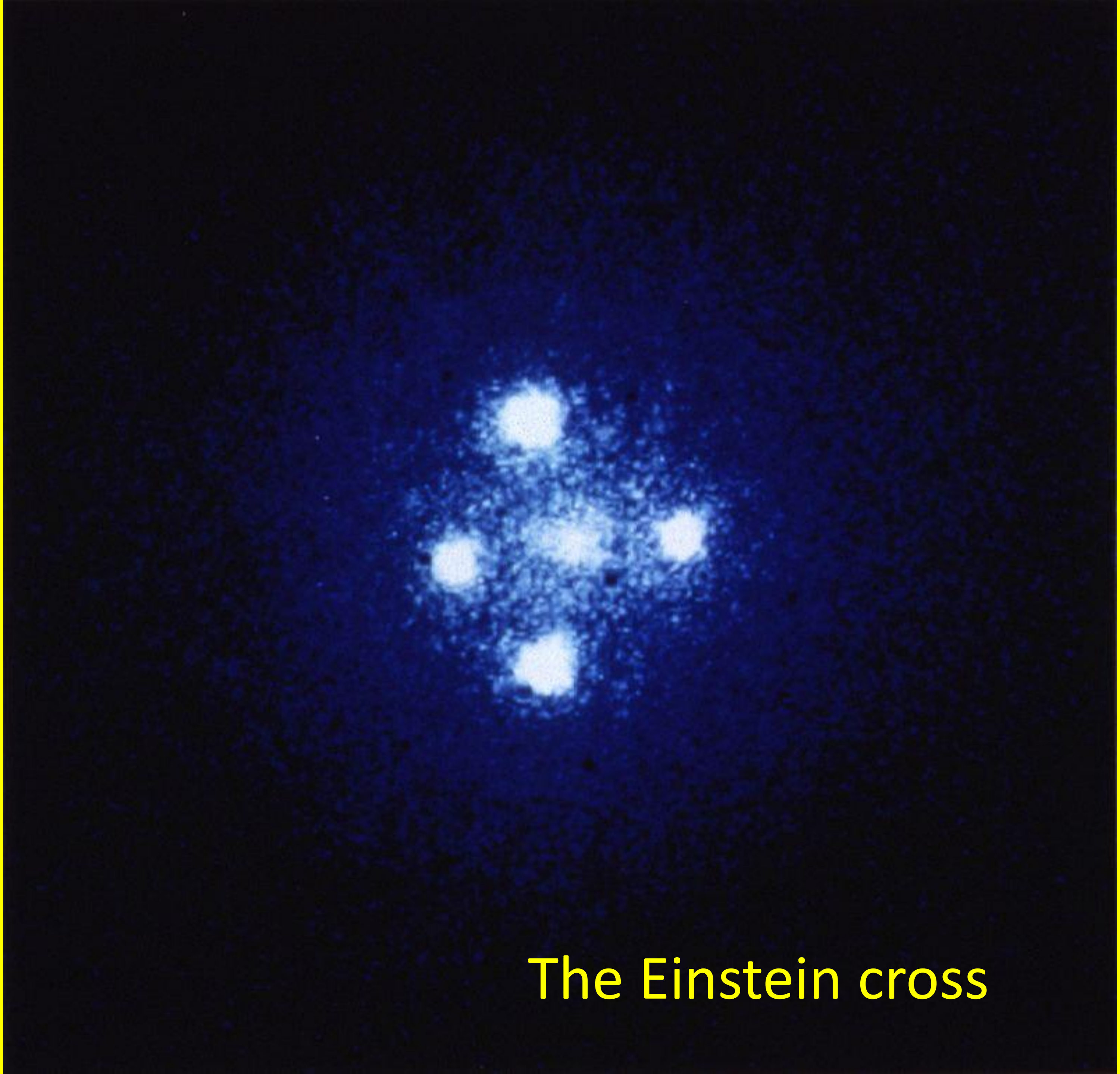
1.75 arcsec
Apparent direction to star
Deflected path of light from star
Light from star
Earth
Sun
Direction to a star ordinarily

Harvey (1979) compares the 1919 results with those he recovered using modern techniques:

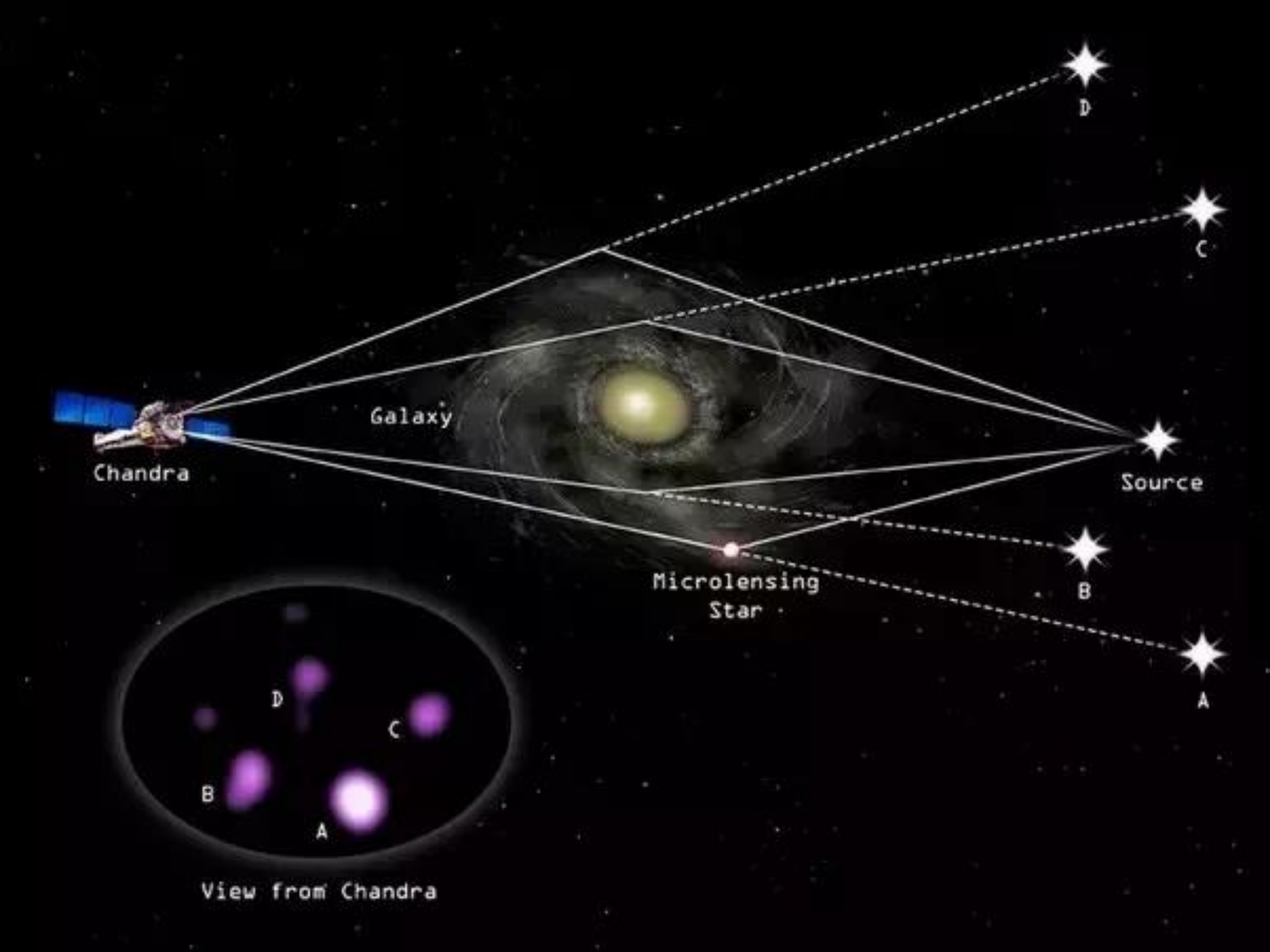
TABLE 1. Gravitational Displacement at the Sun's Limb in Seconds of Arc

Determination	Displacement
Predicted from Einstein's theory	1.75
Four-inch plates reduced by Dyson, Eddington, and Davidson (1920)	1.98 ± 0.18
Four-inch plates measured on the Zeiss	1.90 ± 0.11
Astrographic plates reduced by Dyson, Eddington, and Davidson (1920)	0.93
Astrographic plates measured on the Zeiss	1.55 ± 0.34


Gravitational redshift



The Einstein cross


D
C
Galaxy
Chandra
Source
Microlensing
Star
B
A
D
C
B
A
View from Chandra

How to find a solution of the Einstein equations ?

Specify the metric tensor

$$g_{ab}(x^c)$$

Calculate the inverse metric tensor

$$g^{ab}\,, \;\; g^{ab} \cdot g_{bc} = \delta^a{}_c$$

Calculate the Christoffel symbols

$$\Gamma^d_{ab} = \frac{1}{2} g^{dc} (g_{ac,b} + g_{bc,a} - g_{ab,c})$$

Calculate the Ricci tensor and Ricci scalar

$$R_{ij}, \;\; R = R_{ij} \cdot g^{ij}$$

Specify the energy-momentum tensor

$$T_{ij}$$

Write and solve the Einstein equations.

Relativistics cosmological models

Robertson and Walker showed that the line element of a 3 dimensional homogeneous and isotropic space can be always reduced to the following form:

$$dl^2 = \frac{dr^2}{1 - k\,r^2} + r^2\,(d\Theta^2 + sin^2\Theta\ d\phi^2)$$

where

$$k = \begin{cases} +1 & \text{spherical space} \\ 0 & \text{flat space} \\ -1 & \text{hyperbolic space} \end{cases}$$

The Friedman model also known as Friedman-Lemaitre-Robertson-Walker model:

$$ds^2 = c^2 dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right)$$

Propagation of light in the FLRW model

Let us consider radial rays, Θ = const, and ϕ = const.

Light propagates along null geodesics ds = 0

$$c^2dt^2 - R^2(t)(\frac{dr^2}{1-kr^2}) = 0$$

$$c\int_{t_e}^{t_o} \frac{dt}{R(t)} = \int_0^{r_o} \frac{dr}{\sqrt{1-kr^2}}\,,$$

$$\int_0^{r_o} \frac{dr}{\sqrt{1-kr^2}} = F(r_o) = \begin{cases} \arcsin r_o, & \mathrm{k} = +1\,, \\ r_o, & \mathrm{k} = 0\,, \\ \operatorname{arsinh} r_o, & \mathrm{k} = -\,1\,. \end{cases}$$

$$c\int_{t_e+\delta t_e}^{t_o+\delta t_o} \frac{dt}{R(t)} = F(r_o)$$

$$\frac{\delta t_e}{R(t_e)} = \frac{\delta t_o}{R(t_o)}\,,$$

$$\frac{\delta t_o}{\delta t_e} = \frac{\nu_e}{\nu_0} = \frac{\lambda_o}{\lambda_e} = \frac{R(t_o)}{R(t_e)}\,,$$

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e}\,,\;\; z = \frac{R(t_o)}{R(t_e)} - 1\,,$$

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e}\,, \;\; z = \frac{R(t_o)}{R(t_e)} - 1\,,$$

$$z = \frac{R(t_e + \Delta t)}{R(t_e)} - 1 = \frac{\dot{R}(t_e)}{R(t_e)}\Delta t\,,$$

$$\text{but } \Delta t = r/c, \text{ so } z = \frac{\dot{R}(t_e)}{R(t_e)}\frac{r}{c} = \frac{v}{c}\,,$$

$$\text{for } \tfrac{v}{c} << 1\,.$$

Consequences of the energy-momentum conservation law

$$T^{ij} = (\varepsilon + p)u^i u^j - pg^{ij}$$

$$T^{ij}{}_{;j} = 0$$

$$((\varepsilon + p)u^j)_{;j}u^i + (\varepsilon + p)u^j u^i{}_{;j} - p_{,j}g^{ij} = 0$$

multiplying by u_i, we get

$$((\varepsilon + p)u^j)_{;j} - p_{,j}u^j = 0\,; \;\; \text{since } u^j u^i{}_{;j} u_i = 0$$

$$(\varepsilon + p)u^j u^i{}_{;j} - p_{,j}(g^{ij} - u^i u^j) = 0$$

in homogeneous and isotropic universe p = p(t) $\rightarrow p_{,j}(g^{ij} - u^i u^j) = 0$ so

$u^i{}_{;j}u^j = 0 \rightarrow$ particles move along geodesics.

$$u^i{}_{;j}u^j = 0 \rightarrow \text{particles move along geodesics.}$$

$$((\varepsilon + p)u^j)_{;j} - p_{,j}u^j = 0\,,$$

let us consider simple case - the Universe filled in with pressureless gas

$$\text{then } \ (\varepsilon u^j)_{;j} = 0 \rightarrow$$

$$\frac{1}{\sqrt{-g}}\frac{\delta}{\delta t}(\sqrt{-g}\varepsilon) = 0\,, \quad g = Det|g_{ij}|\,,$$

$$\frac{\delta}{\delta t}(R^3\varepsilon) = 0 \rightarrow \varepsilon R^3 = \text{const}$$